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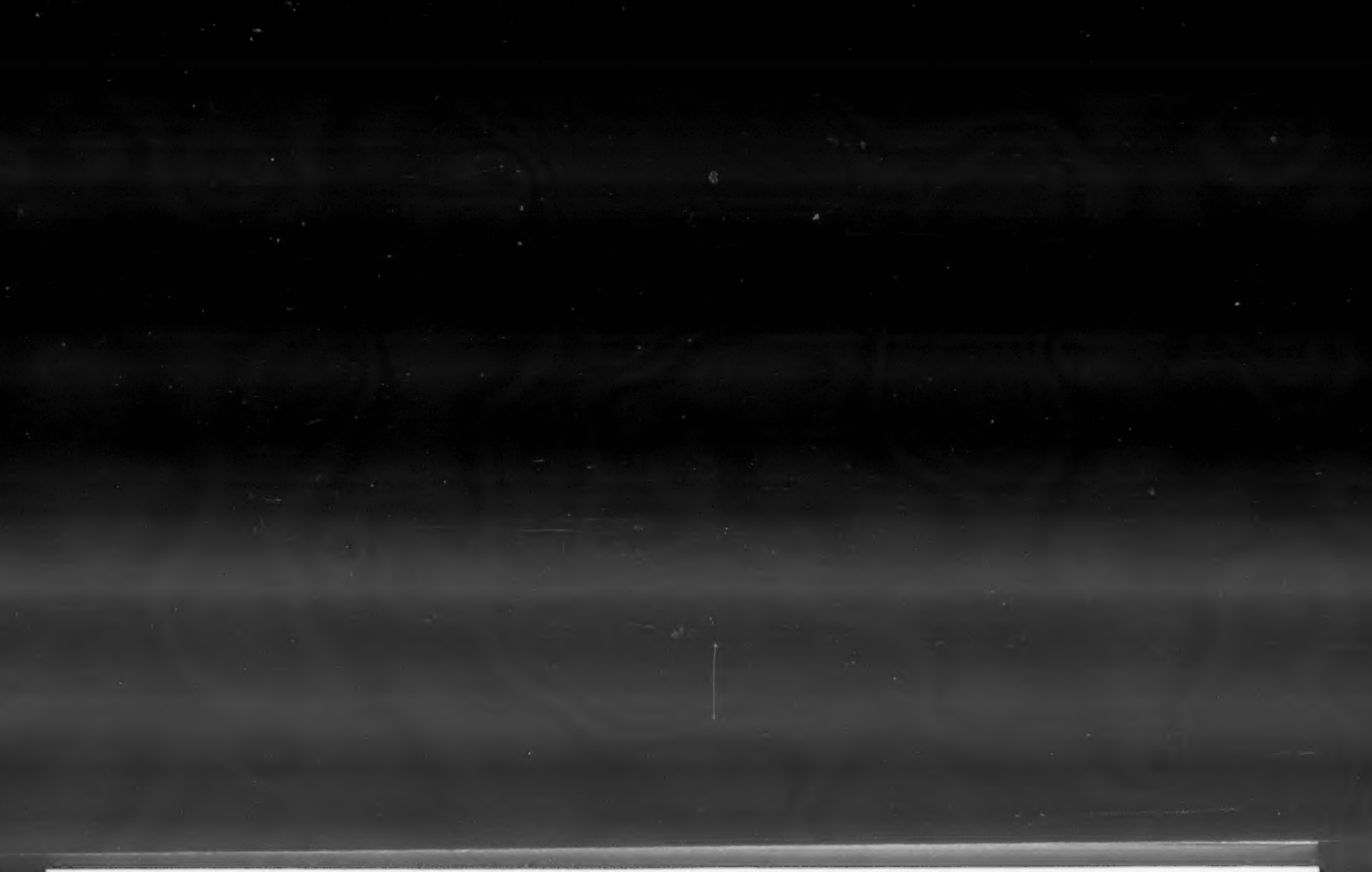

ON THE REAL ROTATION GROUP

By J. C. HOWARTH (*Manchester*)

[Received 25 February 1956]

Introduction

THIS paper deals with the problem of expressing any real m-dimensional rotation matrix (that is, an orthogonal matrix of determinant $+1$) uniquely as a product of $\frac{1}{2}m(m-1)$ rotation matrices of a special type. Although the problem is well known, minor errors have occurred in the statement of the theorem on two previous occasions, first by A. Hurwitz (**1**), then by I. Schur (**2**).

I would like here to express my sincere thanks to Dr. W. Ledermann for his valuable help and encouragement throughout the preparation of this paper.

Notation

If ϕ is any angle and i any integer in the range $1 \leqslant i \leqslant m-1$, denote by $E_i(\phi)$ the matrix of the transformation

$$x_i' = x_i \cos\phi + x_{i+1} \sin\phi,$$
$$x_{i+1}' = -x_i \sin\phi + x_{i+1} \cos\phi,$$
$$x_j' = x_j \quad (j \neq i, i+1).$$

Note that $E_i(\phi)$ is always a rotation matrix and satisfies the relation

$$E_i(\phi+\psi) = E_i(\phi)E_i(\psi).$$

THEOREM. *Let* P_r $(r = 1, 2, \ldots, m-1)$ *be a matrix of the type*

$$P_r = E_{m-r}(\phi_{r-1,r})E_{m-r+1}(\phi_{r-2,r})\ldots E_{m-1}(\phi_{0,r}),$$

where

(i) $0 \leqslant \phi_{r-1,r} < 2\pi$,

$0 \leqslant \phi_{s,r} < \pi \quad (0 \leqslant s \leqslant r-2)$,

(ii) $\phi_{s-1,r} = \phi_{s-2,r} = \ldots = \phi_{0,r} = 0$

if $\phi_{s,r} = 0$ *or* π $(0 \leqslant s \leqslant r-1)$.

Then

(I) *every real m-dimensional rotation matrix A can be expressed as a product*

$$A = P_1 P_2 \ldots P_{m-1},$$

(II) *each of the* $\frac{1}{2}m(m-1)$ *angles* $\phi_{s,r}$ *is uniquely determined by the matrix A.*

Proof. The proof, although rather long, is of an elementary character and will not be given in full. The result is easily verified for the case $m = 2$; assume, therefore, that a rotation matrix A, of dimension m ($\geqslant 3$), is given and that the theorem is true in the dimension $m-1$.

Proof of (I). Suppose that a_k ($1 \leqslant k \leqslant m$) is the last non-zero element in the first row of the matrix A. Thus A has a first row (say)

$$\{a_1 \quad a_2 \quad . \quad . \quad a_k \quad 0 \quad 0 \quad . \quad . \quad 0\},$$

where $a_k \neq 0$. If $k \geqslant 2$, it is easily shown that there is an angle ϕ_{m-k} (say), which is unique in the range $-\pi < \phi_{m-k} < 0$, such that the matrix $AE_{k-1}(\phi_{m-k})$ has first row of the form

$$\{a'_1 \quad a'_2 \quad . \quad . \quad a'_{k-1} \quad 0 \quad 0 \quad . \quad . \quad 0\},$$

where $a'_{k-1} \neq 0$. Repeating this process as often as is necessary, we see that there exist angles (say)

$$\phi_{m-k}, \quad \phi_{m-k+1}, \quad \ldots, \quad \phi_{m-3}, \quad \phi'_{m-2},$$

each uniquely determined in the open interval $(-\pi, 0)$, which have the property that the matrix

$$AE_{k-1}(\phi_{m-k})E_{k-2}(\phi_{m-k+1})\ldots E_2(\phi_{m-3})E_1(\phi'_{m-2})$$

has a first row $\{a\ 0\ 0\ .\ .\ .\ 0\}$, say. The notation ϕ_{m-2} is to be used for an angle now defined from ϕ'_{m-2}, namely,

$$\phi_{m-2} = \begin{cases} \phi'_{m-2} & \text{if } a = 1, \\ \phi'_{m-2} - \pi & \text{if } a = -1. \end{cases}$$

Then $-2\pi < \phi_{m-2} < 0$, but $\phi_{m-2} \neq -\pi$. Now put

$$A_1 = AE_{k-1}(\phi_{m-k})E_{k-2}(\phi_{m-k+1})\ldots E_1(\phi_{m-2}).$$

This concerned $k \geqslant 2$. If $k = 1$, put $A_1 = AE_1(\phi_{m-2})$, where ϕ_{m-2} is 0 or $-\pi$ according as a_1 is $+1$ or -1. Thus, whatever the value of k, A_1 has a first row $\{1\ 0\ 0\ .\ .\ .\ 0\}$. On defining

$$\phi_{m-i,m-1} = \begin{cases} -\phi_{m-i} & (2 \leqslant i \leqslant k), \\ 0 & (k+1 \leqslant i \leqslant m) \end{cases} \qquad (k \geqslant 2),$$

$$\phi_{m-i,m-1} = \begin{cases} -\phi_{m-2} & (i = 2), \\ 0 & (3 \leqslant i \leqslant m) \end{cases} \qquad (k = 1),$$

the proof is completed by applying the induction hypothesis to the matrix A_1.

Proof of (II). Assume that the matrix A has a second representation (say)

$$A = P_1 P_2 \ldots P_{m-1} = Q_1 Q_2 \ldots Q_{m-1},$$

where, for $r = 1, 2, \ldots, m-1$, Q_r is of the same type as P_r. It follows

that the three matrices A, P_{m-1}, Q_{m-1} have the same first row. If $k \geqslant 2$ and

$$Q_{m-1} = E_1(\psi_{m-2})E_2(\psi_{m-3})\ldots E_{m-1}(\psi_0),$$

then it can be shown, by computing the first row of Q_{m-1}, that

$$\psi_{m-k-1} = \psi_{m-k-2} = \ldots = \psi_0 = 0.$$

Thus, the effect of postmultiplying A by either $E_{k-1}(-\phi_{m-k,m-1})$ or $E_{k-1}(-\psi_{m-k})$ is to introduce a zero in place of the element a_k; therefore, from the remarks made during the proof of (I), it follows that

$$\phi_{m-k,m-1} = \psi_{m-k}$$

and repetition of this process gives

$$\phi_{m-i,m-1} = \psi_{m-i} \quad (i = 2, 3,\ldots, k).$$

Similarly, in the case $k = 1$, the corresponding angles are shown to be equal. Hence $P_{m-1} = Q_{m-1}$, so that

$$P_1 P_2 \ldots P_{m-2} = Q_1 Q_2 \ldots Q_{m-2},$$

and the induction hypothesis now completes the proof.

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A FIXED-POINT THEOREM FOR TRANSITION OPERATORS IN AN (L)-SPACE

By F. F. BONSALL (*Newcastle-on-Tyne*)
and G. E. H. REUTER (*Manchester*)

[Received 17 March 1956]

1. LET V be in an abstract (L)-space: namely, a Banach lattice [(**1**) 246] in which the elements of the positive cone V^+ satisfy

$$\|x+y\| = \|x\|+\|y\| \quad (x, y \in V^+). \tag{1}$$

Equivalent assumptions are that V satisfies the axioms I–IX of Kakutani (**4**). It is a well-known and elementary fact that V is a complete lattice in the sense that each sequence $\{x_n\}$ with an upper bound in V has a least upper bound $\sup x_n$ $(1 \leqslant n < \infty)$, and similarly for lower bounds. Also the theorem of Beppo Levi holds in V; if $\{x_n\}$ is an increasing sequence of elements of V^+ such that $\{\|x_n\|\}$ is bounded, then the sequence is bounded above and converges in norm to $\sup x_n$ $(1 \leqslant n < \infty)$. Given a sequence $\{x_n\}$ of elements of V^+ such that $\{\|x_n\|\}$ is bounded, we may therefore define an element $\liminf_{n\to\infty} x_n$ by

$$\liminf_{n\to\infty} x_n = \lim_{k\to\infty} y_k,$$

where

$$y_k = \inf_{k\leqslant n<\infty} x_n.$$

A transition operator in V is a bounded linear operator P in V such that

$$Px \geqslant 0 \quad (x \in V^+), \tag{2}$$

and

$$\|Px\| = \|x\| \quad (x \in V^+). \tag{3}$$

Given an operator P, we denote by Q_n the operator

$$Q_n = (P+P^2+\ldots+P^n)/n \quad (n = 1, 2,\ldots).$$

The main purpose of this note is to prove the following theorem.

THEOREM 1. *Let P be a transition operator in an abstract (L)-space V; let $u \in V^+$, and let $\{n_k\}$ be a strictly increasing sequence of positive integers. If*

$$f = \liminf_{k\to\infty} Q_{n_k} u,$$

then

$$Pf = f.$$

Remarks. (i) We shall prove in § 2 that the theorem holds also for contraction operators: namely, bounded linear operators P in V satisfying (2) and

$$\|Px\| \leqslant \|x\| \quad (x \in V^+), \tag{3'}$$

in place of (3).

(ii) We shall prove in § 3 a result similar to Theorem 1 for $\limsup_{k\to\infty} Q_{n_k} u$, under the assumption that this exists.

(iii) A theorem of Yosida and Kakutani (**5**) on fixed points of transition operators in the space (l) can be deduced from Theorem 1. For other concrete results which have something in common with Theorem 1 but appear to require deeper methods of proof, see Calderón (**2**) and Dowker (**3**). The mean ergodic theorem for abstract (L)-spaces [Kakutani (**4**)] establishes fixed points provided that the sequence $\{Q_n u\}$ has an upper bound in V. Of course Theorem 1, by itself, does not establish the existence of *non-zero* fixed points.

(iv) Our proofs do not use the concrete representation of abstract (L)-spaces.

Proof. It is obvious that for a transition operator P in V the inequalities

$$0 \leqslant Pf \leqslant f$$

imply

$$Pf = f.$$

Thus it suffices to prove the following lemma.

LEMMA. *Let $u \in V^+$, and let P be a bounded linear operator in V satisfying* (2) *and such that for some constant M,*

$$\|P^n u\| \leqslant M \quad (n = 1, 2, \ldots). \tag{4}$$

If $\{n_k\}$ is a strictly increasing sequence of positive integers, and

$$f = \liminf_{k\to\infty} Q_{n_k} u,$$

then

$$Pf \leqslant f.$$

Proof. Let

$$f_i = \inf_{i \leqslant k < \infty} Q_{n_k} u,$$

so that

$$f = \lim_{i\to\infty} f_i,$$

and, by the continuity of P,

$$Pf = \lim_{i\to\infty} Pf_i.$$

We have, for all positive integers n and s,

$$\begin{aligned} PQ_n u &= n^{-1}(P^2 + \ldots + P^{n+1})u \\ &= Q_n u - n^{-1}Pu + n^{-1}P^{n+1}u \leqslant Q_n u + n^{-1}P^{n+1}u, \end{aligned} \tag{5}$$

and $$PQ_n u \leqslant n^{-1}(P+P^2+\ldots+P^{n+s})u = \frac{n+s}{n} Q_{n+s} u. \tag{6}$$

With a fixed integer i, let $\delta_i = n_i^{-\frac{1}{2}}$, and divide the integers j with $j \geqslant i$ into two classes A_i and B_i defined by

$$j \in A_i \quad \text{if } n_j/n_{j-1} \leqslant 1+\delta_i \text{ and } j > i,$$
$$j \in B_i \quad \text{if } n_j/n_{j-1} > 1+\delta_i \text{ or } \quad j = i.$$

Since $f_i \leqslant Q_{n_j} u$ $(j \geqslant i)$, we have

$$Pf_i \leqslant PQ_{n_j} u \quad (j \geqslant i). \tag{7}$$

If $j \in A_i$, then, by (6) and (7),

$$Pf_i \leqslant PQ_{n_{j-1}} u \leqslant (n_j/n_{j-1}) Q_{n_j} u \leqslant (1+\delta_i) Q_{n_j} u. \tag{8}$$

If $j \in B_i$, then, by (5) and (7),

$$Pf_i \leqslant PQ_{n_j} u \leqslant Q_{n_j} u + (1/n_j) P^{n_j+1} u. \tag{9}$$

Denoting the subsequence of integers n_j with $j \in B_i$ by $(m_0, m_1, \ldots)$, we have $m_k = n_j$, say, and

$$m_k > (1+\delta_i) n_{j-1} \geqslant (1+\delta_i) m_{k-1} \quad (k \geqslant 1).$$

Thus $$\sum_{j \in B_i} \frac{1}{n_j} < \frac{1}{n_i}\left\{1 + \frac{1}{1+\delta_i} + \frac{1}{(1+\delta_i)^2} + \ldots\right\} < \frac{1+\delta_i}{n_i \delta_i}.$$

This implies that $$\sum_{j \in B_i} \| n_j^{-1} P^{n_j+1} u \|$$

converges, and therefore there exists an element $v_i \in V$ defined by

$$v_i = \sum_{j \in B_i} n_j^{-1} P^{n_j+1} u.$$

Also $$\|v_i\| \leqslant (1+\delta_i) M/(n_i \delta_i) \leqslant 2M n_i^{-\frac{1}{2}}. \tag{10}$$

By (9), $$Pf_i \leqslant Q_{n_j} u + v_i \quad (j \in B_i),$$

and, combining this with (8), we have

$$Pf_i \leqslant (1+\delta_i) Q_{n_j} u + v_i \quad (j \geqslant i).$$

Writing this inequality in the form

$$(1+\delta_i)^{-1}(Pf_i - v_i) \leqslant Q_{n_j} u \quad (j \geqslant i),$$

we see that $$(1+\delta_i)^{-1}(Pf_i - v_i) \leqslant f_i,$$

and therefore $$Pf_i \leqslant (1+n_i^{-\frac{1}{2}}) f_i + v_i. \tag{11}$$

Letting $i \to \infty$ in (11) and using (10), we have

$$Pf \leqslant f.$$

2. We now prove the assertion in Remark (i) above. Let e be the functional defined on V by

$$(e, x) = \|x^+\| - \|x^-\| \quad (x \in V),$$

where $x = x^+ - x^-$ is the decomposition of x into its positive and negative parts. The functional e is linear. To see this [cf. **(1)** 256], note that

$$(x+y)^+ - (x+y)^- = x+y = x^+ - x^- + y^+ - y^-.$$

Rearranging and applying (1), we find that

$$\|(x+y)^+\| + \|x^-\| + \|y^-\| = \|(x+y)^-\| + \|x^+\| + \|y^+\|;$$

and therefore $$(e, x+y) = (e, x) + (e, y).$$

Let V' denote the space $V + \mathbf{R}$ of all ordered pairs $x' = [x, \xi]$ with $x \in V$ and $\xi \in \mathbf{R}$, normed by taking $\|x'\| = \|x\| + |\xi|$, and partially ordered by taking $V'^+ = V^+ + \mathbf{R}^+$. The space V' is also an (L)-space.

Given a contraction operator P in V, we define an operator P' in V' by taking

$$P'x' = [Px, \xi + (e, x - Px)].$$

It is easily verified that P' is a transition operator.

Let $u \in V^+$, and let $\{n_k\}$ be a strictly increasing sequence of integers. We write $u' = [u, 0]$ and $Q'_n = n^{-1}(P' + \ldots + P'^n)$. Then

$$Q'_n x' = [Q_n x, \xi + (e, x - Q_n x)] \quad (x \in V),$$

and therefore $$f' = \liminf_{k\to\infty} Q'_{n_k} u' = [f, \phi],$$

where $f = \liminf_{k\to\infty} Q_{n_k} u$ and $\phi = \liminf (e, u - Q_{n_k} u)$.

By Theorem 1, we have $P'f' = f'$, i.e.

$$[Pf, \phi + (e, f - Pf)] = [f, \phi].$$

Therefore $Pf = f$, and the proof is complete.

3. If $\{x_n\}$ is a sequence of elements of V^+ which has an upper bound, then $\limsup_{n\to\infty} x_n$ exists, defined by

$$\limsup_{n\to\infty} x_n = \lim_{k\to\infty} y_k,$$

where $y_k = \sup_{k \leq n < \infty} x_n$. The boundedness of the sequence $\{\|x_n\|\}$ does not suffice to establish the existence of $\limsup_{n\to\infty} x_n$. Thus, in general, we are not able to assert the existence of $\limsup_{k\to\infty} Q_{n_k} u$; but, if it does exist, it is easily proved to be a fixed point of P.

THEOREM 2. *Let P be a contraction operator in an abstract (L)-space V, let $u \in V^+$, and let $\{n_k\}$ be a strictly increasing sequence of positive integers. If $\{Q_{n_k} u\}$ has an upper bound in V, then*

$$f = \limsup_{k\to\infty} Q_{n_k} u$$

exists and $Pf = f$.

Proof. Let $f_i = \sup_{i \leqslant k < \infty} Q_{n_k} u$. Then

$$\limsup_{k \to \infty} Q_{n_k} u = \lim_{i \to \infty} f_i = f \text{ (say)}$$

and $Pf = \lim_{i \to \infty} Pf_i$. Since $f_i \geqslant Q_{n_k} u$ $(k \geqslant i)$, we have

$$Pf_i \geqslant PQ_{n_k} u \quad (k \geqslant i). \tag{12}$$

Also $$PQ_n u = Q_n u - n^{-1} Pu + n^{-1} P^{n+1} u \geqslant Q_n u - n^{-1} Pu,$$

so that

$$PQ_{n_k} u \geqslant Q_{n_k} u - n_k^{-1} Pu \geqslant Q_{n_k} u - n_i^{-1} Pu \quad (k \geqslant i).$$

Combining this with (12) we have

$$Pf_i \geqslant Q_{n_k} u - n_i^{-1} Pu \quad (k \geqslant i),$$

and therefore $$Pf_i \geqslant f_i - n_i^{-1} Pu.$$

Letting $i \to \infty$, we obtain $Pf \geqslant f$; and, since $\| Pf \| \leqslant \| f \|$, this completes the proof.

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THE GROUPS $\pi_r(V_{n,m})$ (I)

By G. F. PAECHTER (*Oxford*)

[Received 23 March 1956]

Introduction

THE points of a Stiefel manifold $V_{n,m}$ are ordered sets of m mutually orthogonal unit vectors in euclidean n-space R^n. $V_{n,m}$ is topologized as a subset of R^{nm}.

In this sequence of papers I calculate the homotopy groups $\pi_{k+p}(V_{k+m,m})$ for all k when $0 \leqslant p \leqslant 5$ (except for an ambiguity in the cases $p = 5$, $k = 6$, $m \geqslant 5$), for $k = 1$ and 2 when $p = 6$, and for $k = 1$ when $p = 7$. These results are collected in the following tables, wherein $\pi^p_{k,m}$ denotes $\pi_{k+p}(V_{k+m,m})$, Z_q a cyclic group of order q, and $+$ direct summation. Also $s > 0$.† As each group is calculated in the text, I have specified generators in terms of elements of the homotopy groups of spheres, whose structure is assumed to be well known in the relevant cases.

TABLES FOR $\pi^p_{k,m}$

(a): $p = 0$

	$k=1$	2	$4s-1$	$4s+1$	$4s$	$4s+2$
$m=1$	Z_∞	Z_∞	Z_∞	Z_∞	Z_∞	Z_∞
$m\geqslant 2$	Z_2	Z_∞	Z_2	Z_2	Z_∞	Z_∞

(b): $p = 1$

$m=1$	0	Z_∞	Z_2	Z_2	Z_2	Z_2
$m=2$	0	$Z_\infty+Z_\infty$	Z_2	Z_2	Z_2+Z_∞	Z_2+Z_∞
$m\geqslant 3$	0	Z_∞	0	Z_2	Z_2+Z_2	Z_4

(c): $p = 2$

$m=1$	0	Z_2	Z_2	Z_2	Z_2	Z_2
$m=2$	Z_∞	Z_2+Z_2	Z_4	Z_4	Z_2+Z_2	Z_2+Z_2
$m=3$	$Z_\infty+Z_\infty$	Z_2	Z_2+Z_∞	Z_4+Z_∞	Z_2+Z_2	Z_2
$m\geqslant 4$	Z_∞	0	Z_2	Z_8	Z_2+Z_2	0

† Not all these results are new. For $p = 0$ see Stiefel (**18**); for $p = 1$ see J. H. C. Whitehead (**22**, **23**). Using the erroneous announcement by Pontrjagin that $\pi_5(S^3)$ was trivial (**14**), the following cases had been calculated: $p = 2$ by J. H. C. Whitehead, $m = 2$ and $0 \leqslant p \leqslant 2$ by Eckmann (**6**), and $k = 1$ and $0 \leqslant p \leqslant 4$ by Eckmann (**7**), and G. W. Whitehead (**19**), independently. The groups for $k = 1$ are well known for $p \leqslant 4$, e.g. (**17**), and for $5 \leqslant p \leqslant 7$ have been obtained independently by Borel and Serre (**5**, **16**).

$(d): p = 3$

k	$m=1$	$m=2$	$m=3$	$m=4$	$m \geqslant 5$
1	0	Z_2	Z_2+Z_2	Z_2	0
2	Z_2	Z_2+Z_2	Z_2	Z_∞	0
3	Z_{12}	Z_2	Z_2	Z_2	Z_2
4	$Z_\infty+Z_{12}$	$Z_\infty+Z_{12}+Z_2$	$Z_\infty+Z_{12}+Z_4$	$Z_\infty+Z_{12}+Z_4+Z_\infty$	$Z_{12}+Z_4+Z_\infty$
$8s-1$	Z_{24}	Z_2+Z_2	Z_2+Z_2	Z_2+Z_2	Z_2
$8s+3$	Z_{24}	Z_2+Z_2	Z_2+Z_2	Z_2+Z_2	Z_2+Z_2
$4s+1$	Z_{24}	Z_2+Z_2	$Z_2+Z_2+Z_2$	Z_2+Z_2	Z_2
$8s$	Z_{24}	$Z_{24}+Z_2$	$Z_{24}+Z_4$	$Z_{24}+Z_4+Z_\infty$	$Z_{24}+Z_8$
$8s+4$	Z_{24}	$Z_{24}+Z_2$	$Z_{24}+Z_4$	$Z_{24}+Z_4+Z_\infty$	Z_4+Z_{48}
$4s+2$	Z_{24}	$Z_{24}+Z_2$	$Z_{12}+Z_2$	$Z_{12}+Z_\infty$	Z_{12}

$(e): p = 4$

k	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m \geqslant 6$
1	0	Z_2	Z_2+Z_2	Z_2	Z_∞	0
2	Z_{12}	$Z_{12}+Z_{12}$	0	0	0	0
3	Z_2	$Z_\infty+Z_2$	$Z_\infty+Z_4$	Z_4+Z_∞	$Z_4+Z_\infty+Z_\infty$	Z_4+Z_∞
4	Z_2+Z_2	$Z_2+Z_2+Z_{24}$	Z_2+Z_2+ $+Z_2+Z_2$	$Z_2+Z_2+Z_2+$ $+Z_2+Z_2$	Z_2+Z_2+ $+Z_2+Z_2$	$Z_2+Z_2+Z_2$
5	Z_2	Z_2+Z_2	$Z_2+Z_2+Z_2$	Z_2+Z_2	Z_2+Z_∞	Z_2
$8s-1$	0	Z_2	Z_4	Z_8	Z_8+Z_∞	Z_8
$8s+3$	0	Z_2	Z_4	Z_8	Z_8+Z_∞	Z_{16}
$4s+5$	0	Z_2	Z_2+Z_2	Z_2	Z_∞	0
$4(s+1)$	0	Z_{24}	Z_2+Z_2	$Z_2+Z_2+Z_2$	Z_2+Z_2	Z_2
$8s-2$	0	Z_{24}	Z_2	Z_2	Z_2	0
$8s+2$	0	Z_{24}	Z_2	Z_2	Z_2	Z_2

$(f): p = 5$

k	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m \geqslant 7$
1	0	Z_{12}	$Z_{12}+Z_{12}$	0	0	0	0
2	Z_2	Z_2+Z_2	Z_∞	Z_∞	Z_∞	$Z_\infty+Z_\infty$	Z_∞
3	Z_2	Z_2	Z_2+Z_{24}	$Z_2+Z_2+Z_2$	Z_2+Z_2+ $+Z_2+Z_2$	$Z_2+Z_2+Z_2$	Z_2+Z_2
4	Z_2+Z_2	Z_2+Z_2+ $+Z_2$	Z_2+Z_2+ $+Z_2+Z_2$	$Z_2+Z_2+Z_2+$ $+Z_2+Z_2$	Z_2+Z_2+ $+Z_2+Z_2$	Z_2+Z_2+ $+Z_2+Z_\infty$	Z_2+Z_2+ $+Z_2$
5	Z_2	0	Z_{11}	Z_2	Z_2	Z_2	0
6	Z_∞	Z_∞	$Z_\infty+Z_2$	$Z_\infty+Z_4$	$Z_\infty+Z_8$ or Z_4+Z_∞	$Z_\infty+Z_8+Z_\infty$ or $Z_4+Z_\infty+Z_\infty$	$Z_\infty+Z_8$ or Z_4+Z_∞
$4s+3$	0	0	Z_{24}	Z_2+Z_2	$Z_2+Z_2+Z_2$	Z_2+Z_2	Z_2
$8s+1$	0	0	Z_{24}	Z_2	Z_2	Z_2	Z_2
$8s+5$	0	0	Z_{24}	Z_2	Z_2	Z_2	0
$4(s+1)$	0	0	Z_2	Z_2+Z_2	Z_2	Z_∞	0
$8s+6$	0	0	Z_2	Z_4	Z_8	Z_8+Z_∞	Z_8
$8s+2$	0	0	Z_2	Z_4	Z_8	Z_8+Z_∞	Z_{16}

$(g): p = 6$

k	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$	$m \geqslant 8$
1	0	Z_2	Z_2+Z_6	Z_∞	Z_∞	Z_∞	$Z_\infty+Z_\infty$	Z_∞
2	Z_2	Z_2+Z_2	0	Z_{24}	Z_2+Z_2	$Z_2+Z_2+Z_2$	Z_2+Z_2	Z_2

(h): $p = 7$ *and* $k = 1$

$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$	$m=8$	$m \geqslant 9$
0	Z_2	Z_2+Z_2	0	Z_{24}	Z_2+Z_2	$Z_2+Z_2+Z_2$	Z_2+Z_2	Z_2

Whenever possible in the calculations I have used the fact that $V_{n+1,m+1}$ fibres over the n-sphere S^n with fibre $V_{n,m}$ and that this fibring admits an element of section which can be exhibited in a particularly simple form (a generalization of those of Eckmann in (**6**) 15 d and (**7**) 15 a). Thus the calculations of $\pi_r(V_{n,m})$ proceed by induction on m for fixed $k = n-m$ and r, using the exact homotopy sequence for a fibre space. These calculations take place in §§ 4, 5 of this paper and in the subsequent papers—the earlier sections, and the appendixes to all the papers, being devoted to the assembly of the necessary machinery.

In conclusion I would like to take this opportunity of expressing my gratitude to Prof. J. H. C. Whitehead for his guidance and encouragement. I am also deeply indebted to Dr. M. G. Barratt, for his many suggestions in general, and in particular for his collaboration in obtaining the important result in 5.2 c.

1. Various theorems in fibre-space theory

The triple (X, p, B) is to be a fibre-space X, in the sense of Serre (**15**), over the base-space B, with fibre-mapping $p\colon X \to$ onto B. For $b \in B$, $p^{-1}(b)$ is called the *fibre standing* over b. If b_0 is the base point in B, and $A = p^{-1}(b_0)$, then we write the fibring as $X/A \xrightarrow{p} B$. We thus have the well-known exact homotopy sequences (base points omitted)

$$\longrightarrow \pi_r(A) \xrightarrow{i_{r*}} \pi_r(X) \xrightarrow{j_{r*}} \pi_r(X, A) \xrightarrow{\delta_{r*}} \pi_{r-1}(A) \longrightarrow$$
$$\downarrow p'_{r*}$$
$$\pi_r(B)$$

and

$$\longrightarrow \pi_r(A) \xrightarrow{i_{r*}} \pi_r(X) \xrightarrow{p_{r*}} \pi_r(B) \xrightarrow{\Delta_{r*}} \pi_{r-1}(A) \longrightarrow,$$

where i_{r*} and j_{r*} are the natural injection homomorphisms, δ_{r*} the boundary homomorphism, p'_{r*} the isomorphism induced by p, $p_{r*} = p'_{r*}j_{r*}$, and $\Delta_{r*} = \delta_{r*}p'^{-1}_{r*}$.

We say that the fibring $X/A \xrightarrow{p} B$ admits a *cross-section* $\mathfrak{p}$ if there is a map $\mathfrak{p}\colon B \to X$ such that $p\mathfrak{p}\colon B \to B$ is the identity map. Then $\mathfrak{p}$ induces natural homomorphisms $\mathfrak{p}_{r*}\colon \pi_r(B) \to \pi_r(X)$ with $p_{r*}\mathfrak{p}_{r*}$ the identity isomorphism. Since the homotopy groups are abelian for $r > 1$, we have, by the exactness of the second sequence,

THEOREM 1.1. *A necessary condition for $X/A \to B$ to admit a cross-section is that* (a) $i_{r*}^{-1}(0) = 0$ *for* $r > 0$, (b) $\pi_r(X) = i_{r*}\pi_r(A) + \mathfrak{p}_{r*}\pi_r(B)$ *for* $r > 1$.

Now let B be a sphere S^n. Then we have

THEOREM 1.2. *A necessary and sufficient condition that $X/A \to S^n$ admit a cross-section is that $i_{n-1*}^{-1}(0) = 0$.*

For a proof see (6), Theorem 11.

An *element of section* (Schnittelement) is a map $t' : (E^n, \dot{E}^n) \to (X, A)$ such that $T = pt' : (E^n, \dot{E}^n) \to (S^n, s)$, where $s = p(A)$, is topological on $E^n - \dot{E}^n$. Let the orientation be such that T is of degree 1. Let $t : S^{n-1} \to A$ be the map defined by $t' \mid \dot{E}^n$. Then, for each r, t induces a homomorphism $t_* : \pi_{r-1}(S^{n-1}) \to \pi_{r-1}(A)$. Then we have

THEOREM 1.3. $t_*\pi_{r-1}(S^{n-1}) = i_{r-1*}^{-1}(0)$ *when* $r < 2n-1$ ($r = 2n-1$ *if* n *is odd*).

The proof of this theorem follows from the fact that

$$\mathfrak{E}\pi_{r-1}(S^{n-1}) = \pi_r(S^n) \quad \text{when } r < 2n-1$$

($r = 2n-1$ if n is odd), where $\mathfrak{E}$ denotes the Freudenthal suspension homomorphism (9), and the following theorem:

THEOREM 1.4. $t_* = \Delta_{r*}\mathfrak{E} : \pi_{r-1}(S^{n-1}) \to \pi_{r-1}(A)$.

For the proof of this theorem see (6) 172 et seq. In consequence of 1.3 we have

COROLLARY 1.5. *If* $r = n$, *then* $\{t\}$ *generates* $i_{n-1*}^{-1}(0)$.†

2. Application to the Stiefel manifolds $V_{n,m}$

2.1. *General properties of* $V_{n,m}$. The points z of $V_{n,m}$ are ordered sets of m mutually orthogonal unit vectors $(z_1, z_2, \ldots, z_m)$ in R^n (Euclidean n-space). Thus $V_{n,1} = S^{n-1}$, the unit sphere in R^n. We topologize $V_{n,m}$ as a subset of R^{nm}. In (6) Eckmann showed that $V_{n,m}$ is a fibre-space over $V_{n,k}$, with fibre homeomorphic to $V_{n-k,m-k}$, and fibre mapping $p_{n,k} : V_{n,m} \to V_{n,k}$ given by $p_{n,k}(z_1, z_2, \ldots, z_m) = (z_1, z_2, \ldots, z_k)$.

Let a point z of $V_{n,m}$ be represented by the matrix $\|v_{i,j}\|$, having as its rows the ordered unit vectors of z. As base point in $V_{r,s}$ we take the point $v_0 = \|v_{i,j}\|$, where $v_{i,j} = -\delta_{i,s-j+1}$. We define the identical map $i_{n-k,k} : V_{n-k,m-k} \to V_{n,m}$ as

$$i_{n-k,k}(\|w_{i,j}\|) = (\|v_{i,j}\|) \quad (\|w_{i,j}\| \in V_{n-k,m-k};\ \|v_{i,j}\| \in V_{n,m}),$$

$$v_{i,j} = \begin{cases} -\delta_{i,n-j+1} & (i \leqslant k), \\ w_{i-k,j} & (i > k;\ j \leqslant m-k), \\ 0 & (i > k;\ j > m-k). \end{cases}$$

† $\{k\}$ denotes the homotopy class of a map k.

Then we see that $i_{n-k,k}(V_{n-k,m-k})$ is the fibre standing over the base point v_0 of $V_{n,k}$. Further, for $s \leqslant k$, $i_{n-s,s}$ induces the fibre-preserving map

$$\bar{i}_{n-s,s}\colon (V_{n-s,m-s}, p_{n-s,k-s}, V_{n-s,k-s}) \to (V_{n,m}, p_{n,k}, V_{n,k}),$$

the induced maps being $i_{n-s,s}$ on the base space $V_{n-s,k-s}$, and the identity $i_{n-k,0}$ on the fibre $V_{n-k,m-k}$. Similarly, for $k \leqslant s$, $p_{n,s}$ induces the fibre-preserving map

$$\bar{p}_{n,s}\colon (V_{n,m}, p_{n,k}, V_{n,k}) \to (V_{n,s}, p_{n,k}, V_{n,k}),$$

the induced maps being $p_{n,k}$ on the base space $V_{n,k}$, and $p_{n-k,s-k}$ on the fibre $V_{n-k,m-k}$. It follows that the homomorphisms of homotopy groups induced by the injections $i_{r,s}$ and the projections $p_{n,k}$ commute with each other and with the homotopy boundary homomorphism whenever the composition makes sense.

2.2. *Construction of an element of section for* $V_{n+1,m+1}/V_{n,m} \to S^n$. Let E^n be given by the equations

$$\sum_{i=0}^{n} x_i^2 = 1, \qquad x_n \geqslant 0,$$

in R^{n+1}, $S^{n-1} = \dot{E}^n$ being its intersection with $x_n = 0$. Then we define

$$t'_{n+1,m+1}\colon (E^n, \dot{E}^n) \to (V_{n+1,m+1}, V_{n,m})$$

by

$$t'_{n+1,m+1}(x_0, x_1, \ldots, x_n) = v_{i,j}$$

in $V_{n+1,m+1}$, where

$$v_{i,j} = 2x_{n+1-i}\,x_{j-1} - \delta_{n+1-i,j-1} \quad (i = 1, \ldots, m+1;\ j = 1, \ldots, n+1).$$

It is easily seen that $t'_{n+1,m+1}$ has all the properties required of an element of section, and we observe that $t_{n+1,m+1}\colon S^{n-1} \to V_{n,m}$ is given by

$$t_{n+1,m+1}(x_0, x_1, \ldots, x_{n-1}) = u_{i,j}$$

in $V_{n,m}$, where

$$u_{i,j} = 2x_{n-i}\,x_{j-1} - \delta_{n-i,j-1} \quad (i = 1, \ldots, m;\ j = 1, \ldots, n).$$

2.3. *Properties of* $t_{n+1,m+1}$.

(*a*) $t_{n+1,m+1}$ is a symmetric map; i.e. $tx = tx^*$, where x and x^* are diametrically opposite points of S^{n-1};

(*b*) (i) $p_{n,k}\,t_{n+1,m+1} = t_{n+1,k+1}\colon S^{n-1} \to V_{n,k}$.

Thus (ii) $p_{n,1}\,t_{n+1,k+1} = t_{n+1,2}\colon S^{n-1} \to S^{n-1}$,

which is of degree 2 if n is even, and of zero degree if n is odd. Geometrically this is the map which assigns to every point s of S^{n-1} the reflection of $(0, 0, \ldots, -1)$ in the $n-1$ flat through the centre of S^{n-1} which is orthogonal to the line joining s to the centre. It maps the S^{n-2} given by $x_{n-1} = 0$ onto the point $(0, \ldots, 0, -1)$.

Note also that

(iii) $t_{n+1,m+1} \mid S^{n-i}$

$$= i_{n-i+1,i-1} t_{n-i+2,m-i+2} \colon S^{n-i} \to i_{n-i+1,i-1}(V_{n-i+1,m-i+1}),$$

where S^{n-i} is the intersection of S^{n-1} with

$$x_{n-1} = x_{n-2} = \ldots = x_{n-i+1} = 0.$$

(*c*) If $u_r \colon S^r \to P^r$ is the identification map which identifies x and x^* in S^r, P^r being the real r-dimensional projective space, then

$$p_{n,1} t'_{n+1,m+1} u_{n-1}^{-1} \colon P^{n-1} \to S^{n-1}$$

is single-valued and so continuous (**24**), and of degree 1 (mod 2) and therefore essential. Further, it maps the P^{n-2} given by $x_{n-1} = 0$ onto the point $s = (0,\ldots,0,-1)$, and is topological on $(P^{n-1} - P^{n-2})$. We choose the orientation of the latter such that this map, when restricted to $(P^{n-1} - P^{n-2})$, is of degree 1.

LEMMA 2.3 (*d*). *The image of S^{n-1} under $t_{n+1,m+1}$ in $V_{n,m}$ is the homeomorphic image of a P_{k-1}^{n-1}, where $k = n-m$,† and P_{k-1}^{n-1} is the projective space P^{n-1} with a subspace P^{k-1} shrunk to a point.*

Proof. Let

$$\phi_{n+1,m+1} = t_{n+1,m+1} u_{n-1}^{-1} \colon P^{n-1} \to V_{n,m}.$$

Then we see that $\phi_{n+1,m+1}$ maps the P^{k-1} which is the intersection of P^{n-1} with $x_{n-1} = x_{n-2} = \ldots = x_k = 0$ onto a point, while it is one-to-one on $P^{n-1} - P^{k-1}$. If $w_{r,s} \colon P^r \to P_s^r$ is the identification map which shrinks the subspace P^s to a point p (P_s^r being given the identification topology), then

$$\psi_{n+1,m+1} = \phi_{n+1,m+1} w_{n-1,m-1}^{-1} \colon P_{k-1}^{n-1} \to V_{n,m}$$

is single-valued and so continuous, and one-to-one. But it is a map of a compact space into a Hausdorff space: that is, it is a homeomorphism 'into'. Note that

$$t_{n+1,m+1} = \psi_{n+1,m+1} w_{n-1,k-1} u_{n-1} \colon S^{n-1} \to V_{n,m},$$

so that t and ψ do map onto the same space. Hereafter I shall refer to this image as 'the P_{k-1}^{n-1} imbedded in $V_{n,m}$'. We also see that § 2.3 (*b*) (iii) implies a similar relation for $\psi_{n+1\,m+1} \mid P_{k-i}^{n-i}$.

Further note that, if E^n is the hemisphere of S^n given by $x_n \geqslant 0$, then

$$t_{n+2,m+2} \mid E^n = t'_{n+1,m+1} \colon (E^n, \dot{E}^n) \to (V_{n+1,m+1}, V_{n,m}).$$

Since restricting $t_{n+2,m+2}$ to E^n has no effect on $\phi_{n+2,m+2}$, we see that the image of $t'_{n+1,m+1}$ is the P_{k-1}^n imbedded in $V_{n+1,m+1}$. Let us consider P^n

† This was proved originally explicitly in (**22**) 250.

and P^n_{k-1} as CW complexes. (I shall drop subscripts of maps whenever ambiguity cannot arise.) Then we have from 2.3 (c) that

$$p_{n+1,1}\phi_{n+2,m+2} = p_{n+1,1}t_{n+2,m+2}u_n^{-1}\colon (P^n, P^{n-1}) \to (S^n, s)$$

is topological and of degree one when restricted to $(P^n - P^{n-1})$. Since the same is true of

$$w_{n,k-1}\colon (P^n, P^{k-1}) \to (P^n_{k-1}, p),$$

we have that $\quad p\psi = p\phi w^{-1}\colon (P^n_{k-1}, P^{n-1}_{k-1}) \to (S^n, s)$

is topological and of degree one when restricted to $P^n_{k-1} - P^{n-1}_{k-1}$.

But $\quad p\psi\psi^{-1}t' = p_{n+1,1}t'_{n+1,m+1}\colon (E^n, \dot{E}^n) \to (S^n, s),$

which is topological and of degree one when restricted to $(E^n - \dot{E}^n)$. Thus

$$g = \psi^{-1}t'\colon (E^n, \dot{E}^n) \to (P^n_{k-1}, P^{n-1}_{k-1})$$

is a characteristic map for the n-cell of P^n_{k-1}. Let

$$\delta'_{r*}\colon \pi_r(E^n, \dot{E}^n) \to \pi_{r-1}(S^{n-1})$$

be the boundary homomorphism. Then P. J. Hilton showed [(**10**), proof of Theorem 1.1] that

$$(p\psi)_{r*}g_{r*}\delta'^{-1}_{r*} = \mathfrak{E}\colon \pi_{r-1}(S^{n-1}) \to \pi_r(S^n),$$

i.e. $$p'_{r*}\psi_{r*}g_{r*}\delta'^{-1}_{r*} = \mathfrak{E}.$$

But (i) p'_{r*} and δ'^{-1}_{r*} are always isomorphisms 'onto',

(ii) g_{r*} is 'onto' if $1 < r < n+k-1$, by (**25**) Theorem 1, since $\pi_r(P^{n-1}_{k-1}) = 0$ for $r < k$.

Hence we have the following lemmas:

LEMMA 2.3 (e). *If* $\pi_r(S^n) = \mathfrak{E}\pi_{r-1}(S^{n-1})$, *then*

$$\psi_{n+2,m+2*}\colon \pi_r(P^n_{k-1}, P^{n-1}_{k-1}) \to \pi_r(V_{n+1,m+1}, V_{n,m})$$

is 'onto'.

LEMMA 2.3 (f). *If* $1 < r < n+k-1$, *and* $\mathfrak{E}\colon \pi_{r-1}(S^{n-1}) \approx \pi_r(S^n)$, *then*

$$\psi_{n+2,m+2*}\colon \pi_r(P^n_{k-1}, P^{n-1}_{k-1}) \approx \pi_r(V_{n+1,m+1}, V_{n,m}).$$

By use of induction on m and the 'five' lemma these lemmas lead to a proof of

THEOREM 2.3 (g). *If* $r < 2k$ $(k = n-m > 0)$,

then $\quad \psi_{n+1,m+1*}\colon \pi_r(P^{n-1}_{k-1}) \approx \pi_r(V_{n,m}),$

and $\quad \psi_{n+1,m+1*}\colon \pi_{2k}(P^{n-1}_{k-1}) \to \pi_{2k}(V_{n,m})$

is 'onto'.

I omit this proof. A different proof can be found in (**22**), Theorem 3.

3.1. *Notation and properties of homotopy groups.* The following conventions will be used throughout.

$\pi^p_{k,m}$ denotes the group $\pi_{p+k}(V_{k+m,m})$, $\mathfrak{E}$ the suspension functor. If $f\colon (E^r, \dot{E}^r, e_0) \to (X, A, a_0)$, where A may equal a_0, then $\{f\}$ denotes the homotopy class of f in $\pi_r(X, A, a_0)$. As a rule base points will be omitted. $[a, b]$ in $\pi_{p+q-1}(X)$ denotes the Whitehead product of a in $\pi_p(X)$ and b in $\pi_q(X)$ (**21**). It will sometimes be used also with a and b as maps instead of classes. A similar notation will be used for generalized Whitehead products.

If $\pi_r(S^n)$ is a cyclic group, then $h_{n,r}\colon S^r \to S^n$ will be such that $\{h_{n,r}\}$ generates $\pi_r(S^n)$; $h_{r,r}$ will be of degree one, $h_{2,3}$ of Hopf invariant one, and $\{h_{3,6}\}$ will be the Blakers–Massey element. Also $\bar{p}$ will denote the Hopf fibre map $: S^7 \to S^4$, of invariant one. Thus, for instance, we write generators of $\pi_7(S^4)$, $= Z_\infty + Z_{12}$, as $\{\bar{p}h_{7,7}\}$ and $\mathfrak{E}\{h_{3,6}\}$. It will be assumed that the reader is familiar with the properties of $\pi_{n+k}(S^n)$ for $k \leqslant 5$: that is, with their structure, their behaviour under $\mathfrak{E}$, and the values of compositions $\{h_{n,s}h_{s,r}\}$ and Whitehead products.

Let $\{h\} \in \mathfrak{E}\pi_{r-1}(S^{n-1})$; $\{h'\} = \delta_*^{-1}\{h\}$, where

$$\delta_*\colon \pi_{r+1}(E^{n+1}, \dot{E}^{n+1}) \approx \pi_r(S^n).$$

Then, if A is arcwise connected, we have

LEMMA 3.1 (*a*). *$\{h\}$ and $\{h'\}$ induce homomorphisms $h^*\colon \pi_n(A) \to \pi_r(A)$ and $h'^*\colon \pi_{n+1}(X, A) \to \pi_{r+1}(X, A)$;*

(*b*) *these homomorphisms commute with $\mathfrak{E}$, a fibre mapping, a cross-section, and the homomorphisms of the exact homotopy sequences of* § 1.

The proof is straightforward and therefore omitted. When an h^* appears in the sequel, it will always be a homomorphism, and this in virtue of the above lemma unless otherwise stated. Note also that

$$h^*\{k\} = \{kh\} = k_*\{h\}.$$

3.2. *A theorem on suspension.* Let X be a CW complex, X^p its p-section. Let X be $(k-1)$-connected, where $k > 1$. Then we have the following theorems:

THEOREM 3.2 (A).

$$(a)\ \ \mathfrak{E}\colon \pi_{r-1}(X) \to \pi_r(\mathfrak{E}X)$$

is an isomorphism for $r < 2k$.

(*b*) *When $r = 2k$, $\mathfrak{E}$ is 'onto' and $\mathfrak{E}^{-1}(0)$ is generated by the products $[\alpha, \beta]$, where α and β are any elements in $\pi_k(X)$.*

THEOREM 3.2 (B).

$$\mathfrak{E}\colon \pi_{r-1}(X^{p+1}, X^p) \to \pi_r(\mathfrak{E}X^{p+1}, \mathfrak{E}X^p)$$

is an isomorphism when $2 < r < p+k+1$, *and 'onto' when* $r = p+k+1$.

Proof of Theorem 3.2 (A). Let X_1 and X_2 be two cones on X intersecting in X, so that $X_1 \cup X_2 = \mathfrak{E}X$, and $X_1 \cap X_2 = X$. Then consider the diagram

$$\begin{array}{ccccccccc} \to \pi_{r+1}(\mathfrak{E}X; X_1, X_2) & \xrightarrow{\delta_*} & \pi_r(X_1, X) & \xrightarrow{i_*} & \pi_r(\mathfrak{E}X, X_2) & \xrightarrow{j_*} & \pi_r(\mathfrak{E}X; X_1, X_2) \to \\ & & \downarrow{\scriptstyle \delta_{1*}} & & \uparrow{\scriptstyle i''_*} & & \\ & & \pi_{r-1}(X) & \xrightarrow{\mathfrak{E}} & \pi_r(\mathfrak{E}X). & & \end{array}$$

By Theorem 1 of (**13**) the sequence is exact. Since X_1 and X_2 are contractible, δ_{1*} and i''_* are isomorphisms. Also, by an argument similar to that on p. 375 of (**13**), the diagram is commutative. Hence i_* and $\mathfrak{E}$ are equivalent. Since (X_1, X) and (X_2, X) are both k-connected,

$$\pi_r(\mathfrak{E}X; X_1, X_2) = 0 \quad \text{for } r \leqslant 2k$$

by Theorem 1 of (**3**) (II). This proves (*a*) and the first part of (*b*).

Now by Theorem 1 of (**3**) (III), $\pi_{2k+1}(\mathfrak{E}X; X_1, X_2)$ is generated by the products $[\alpha', \beta']$ for any elements $\alpha' \in \pi_{k+1}(X_2, X)$ and $\beta' \in \pi_{k+1}(X_1, X)$. Further, if $\delta_{2*}\colon \pi_r(X_2, X) \approx \pi_{r-1}(X)$, then

$$\delta_*[\alpha', \beta'] = -[\delta_{2*}\alpha', \beta'], \quad \text{by 4.3 of (4)},$$

and $$\delta_{1*}[\delta_{2*}\alpha', \beta'] = (-1)^k[\delta_{2*}\alpha', \delta_{1*}\beta'], \quad \text{by 3.5 of (4)}.$$

Putting $\alpha = \delta_{2*}\alpha'$ and $\beta = \delta_{1*}\beta'$, we obtain the last part of (*b*).

Proof of Theorem 3.2 (B). This is omitted. It follows directly from the special case of (A) when X is a bunch of p-spheres having a single common point, and Theorem 1 of (**25**).

Now let Y_2^{k+1} be the space consisting of an S^k to which one $k+1$ cell has been attached by a map ϕ such that $\phi \mid \dot{E}^{k+1} \to S^k$ is of degree 2. Let B_2^{k+2} be the space consisting of Y_2^{k+1} to which one $k+2$ cell has been attached by a map ϕ such that $\phi \mid \dot{E}^{k+2} \to S^k \subset Y_2^{k+1}$ and is essential on S^k. Then Theorem 3.2 (A) yields

COROLLARY 3.2 (C). (*a*) $\mathfrak{E}\colon \pi_{k+2}(Y_2^{k+1}) \approx \pi_{k+3}(Y_2^{k+2})$ $(k \geqslant 3)$;

(*b*) $\mathfrak{E}\colon \pi_{k+2}(B_2^{k+2}) \approx \pi_{k+3}(B_2^{k+3})$ $(k \geqslant 3)$.

(When $k = 3$, all products in $\pi_6(Y_2^4)$ and $\pi_6(B_2^5)$ must be injections of products in $\pi_6(S^3)$, which are all zero.)

3.3. *An algebraic theorem*

Definition. A subgroup $U \subset G$ is *partially division-closed* if and only if for every $g \in G$ such that $mg \in U$, there is a $u \in U$ such that $mu = mg$.

THEOREM 3.3 (A). *Let G be a finitely generated abelian group, U a subgroup. Then a necessary and sufficient condition for U to be a direct summand of G is that U be partially division-closed.*

Proof. (*a*) Let $G = U+V$ and let $mg \in U$, where

$$g = u+v \ (u \in U, \ v \in V).$$

Then $mv = mg-mu \in U$, whence $mv = 0$. Thus $mg = mu$, and U is partially division-closed.

(*b*) Let U be partially division-closed, $V = G-U$, and $f: G \to V$ the natural homomorphism. Since G is a finitely generated abelian group, so is V. Therefore $V = V_1+\ldots+V_n$, where each V_i is cyclic. Let v_i, of order m_i, be a generator of V_i and let $g_i \in f^{-1}v_i$. Then $m_i g_i \in U$ since $m_i v_i = 0$. Therefore $m_i g_i = m_i u_i$ for some $u_i \in U$. Let $g_i' = g_i-u_i$. Then $fg_i' = v_i$ and $m_i g_i' = 0$. Therefore a homomorphism $h: V \to G$, such that $fh = 1$, is defined by $hv_i = g_i'$ for each $i = 1$ to n. Hence, since G is abelian, $G = U+hV$, and the theorem is proved.

Let G be a finitely generated abelian group, $U \subset G$ finite, and, for every prime p, let $N(G,p)$ be the maximal order of elements in the p-component of G.

LEMMA 3.3 (B). *If $U = U_1+\ldots+U_r$, where U_i is cyclic of order*

$$N(G,p_i) > 1,$$

for some prime p_i, then U is partially division-closed in G.

Proof.† If

$$G = G'+G'', \qquad U = U'+U'', \qquad U' \subset G', \qquad U'' \subset G'',$$

and if U' and U'' are partially division-closed in G' and G'', then so obviously is U in G. Therefore we may consider the p-components of G separately. Let X be the p-component of G for some prime p, V that of U, and suppose that $N(G,p) = p^t$. Suppose that $V = V_1+\ldots+V_r$, where V_i is cyclic of order p^t, and let v_i be a generator of V_i. Suppose that $g \in X$ is such that

$$mg = k_1 v_1+\ldots+k_r v_r,$$

where $m = hp^s$, h being prime to p and $s \leqslant t$. Then there exists an h' such that $hh' \equiv 1 \pmod{p^t}$. Also $p^{t-s}mg = 0$, whence every $k_i = l_i p^s$. Thus

$$mg = p^s u = p^s hh'u = m(h'u),$$

† I am indebted to the referee for this simplified version of the original proof.

where $u = l_1 v_1 + \ldots + l_r v_r$. By the above observation, this proves the lemma.

COROLLARY 3.3 (C). *Under the conditions of Lemma* 3.3 (B), U *is a direct summand of* G.

4. Calculation of $\pi^p_{k,m}$: preliminaries

The calculations are based on the examination of the exact sequences associated with the fibrings

$$V_{k+m+1,m+1}/V_{k+m,m} \to S^{k+m}.$$

When these are insufficient, we turn in the first place to those associated with the fibrings

$$V_{k+m+1,m+1}/S^k \to V_{k+m+1,m}.$$

Constant use will be made of Theorem 1.3; because of this frequency it will not be referred to every time it is applied.

I first prove two theorems:

THEOREM 4.1. (*a*) $\pi^p_{k,m} \approx \pi^p_{k,p+2}$ *for* $m \geqslant p+2$.

(*b*) $\pi^p_{k,m} = 0$ *for* $p < 0$ *and all* k *and* m.

Proof. (*a*) Consider the sequence associated with the fibring

$$V_{k+m+1,m+1}/V_{k+m,m} \to S^{k+m} \quad \text{when } r = k+p;$$

i.e. $$\to \pi_{k+p+1}(S^{k+m}) \xrightarrow{\Delta_*} \pi^p_{k,m} \xrightarrow{i_*} \pi^p_{k,m+1} \xrightarrow{p_*} \pi_{k+p}(S^{k+m}) \to.$$

Hence, since $\pi_r(S^{k+m}) = 0$ when $r < k+m$, $\pi^p_{k,m+1} \approx \pi^p_{k,m}$ when $p \leqslant m-2$; and (*a*) follows.

(*b*) By (*a*), $\pi^p_{k,m} \approx \pi^p_{k,1}$ if $p < 0$, i.e. $\approx \pi_{k+p}(S^k) = 0$ since $p < 0$.

THEOREM 4.2. (*a*) $\pi^p_{2,m} \approx \pi^{p+1}_{1,m+1}$ $(p \geqslant 1)$.

(*b*) $\pi^5_{k,m} \approx \pi^4_{k+1,m-1}$ $(k \geqslant 7)$.

Proof. (*a*) Consider the sequence associated with the fibring

$$V_{m+2,m+1}/S^1 \to V_{m+2,m},$$

i.e. $$\to \pi_{p+2}(S^1) \to \pi^{p+1}_{1,m+1} \xrightarrow{p_{2+m,m*}} \pi^p_{2,m} \to \pi_{p+1}(S^1) \to.$$

Hence, since $\pi_r(S^1) = 0$ for $r > 1$, (*a*) follows.

(*b*) Consider the sequence associated with the fibring

$$V_{k+m,m}/S^k \to V_{k+m,m-1}, \quad \text{when } r = k+5,$$

i.e. $$\to \pi_{k+5}(S^k) \to \pi^5_{k,m} \xrightarrow{p_{k+m,m-1*}} \pi^4_{k+1,m-1} \to \pi_{k+4}(S^k) \to.$$

But $\pi_{k+5}(S^k) = 0 = \pi_{k+4}(S^k)$ when $k \geqslant 7$; whence (*b*) follows.

5. Calculation of $\pi^p_{k,2}$

We consider the fibring $V_{k+2,2}/S^k \to S^{k+1}$, and examine the sequence

$$\text{(A)} \quad \to \pi_{k+p+1}(S^{k+1}) \xrightarrow{\Delta_*} \pi_{k+p}(S^k) \xrightarrow{i_{k+p*}} \pi^p_{k,2} \xrightarrow{p_{k+p*}} \pi_{k+p}(S^{k+1}) \to.$$

5.1. $k \equiv 0 \pmod 2$. In this case there is a one-field on S^{k+1} [see (8), (19)] and so the fibring admits a cross-section p. Hence Theorem 1.1 gives

$$\pi^p_{k,2} = i_* \pi_{k+p}(S^k) + \mathrm{p}_* \pi_{k+p}(S^{k+1}).$$

The values of $\pi^p_{k,2}$ for even k are then as shown in the tables.

Note that, by Theorem 1.2 and Corollary 1.5, we have

$$\{t_{k+2,2}\} = 0 \quad \textit{for } k \equiv 0 \pmod 2.$$

5.2. $k \equiv 1 \pmod 2$ *and* $k \geqslant 3$. We have from 2.3 (*b*) that

$$t_{k+2,2} \colon S^k \to S^k$$

is of degree 2. Hence $\{t_{k+2,2}\} = 2\{h_{k,k}\}$. Then

(*a*) When $p = 0$, (A) gives

$$\xrightarrow{p_{k+1*}} \pi_{k+1}(S^{k+1}) \xrightarrow{\Delta_*} \pi_k(S^k) \xrightarrow{i_{k*}} \pi^0_{k,2} \to \pi_k(S^{k+1}) \to$$

i.e.

$$\to Z_\infty \to Z_\infty \to \pi^0_{k,2} \to 0,$$

where $i^{-1}_{k*}(0)$ is generated by $\{t_{k+2,2}\}$: that is, by $\{2h_{k,k}\}$.

Hence $\pi^0_{k,2} = Z_2$, *generated by* $\{i_{k+1,1} h_{k,k}\}$.

Note that $\Delta_*^{-1}(0) = 0$, and hence that p_{k+1*} is trivial.

(*b*) When $p = 1$, (A) gives

$$\xrightarrow{p_{k+2*}} \pi_{k+2}(S^{k+1}) \xrightarrow{\Delta_*} \pi_{k+1}(S^k) \xrightarrow{i_{k+1*}} \pi^1_{k,2} \xrightarrow{p_{k+1*}} \pi_{k+1}(S^{k+1})$$

i.e.

$$\to Z_2 \to Z_2 \to \pi^1_{k,2} \to 0$$

since p_{k+1*} is trivial by (*a*). Also

$$i^{-1}_{k+1*}(0) = t_{k+2,2*}\, \pi_{k+1}(S^k),$$

which is generated by

$$h^*_{k,k+1}\{t_{k+2,2}\} = h^*_{k,k+1}\, 2\{h_{k,k}\} = 2h^*_{k,k+1}\{h_{k,k}\} = 2\{h_{k+1,k}\} = 0.$$

Hence $i^{-1}_{k+1*}(0) = 0$, and $\pi^1_{k,2} = Z_2$, *generated by* $\{i_{k+1,1} h_{k,k+1}\}$.

Note that thus Δ_* is trivial, and hence p_{k+2*} is 'onto'.

(*c*) When $p = 2$, (A) gives

$$\xrightarrow{p_{k+3*}} \pi_{k+3}(S^{k+1}) \xrightarrow{\Delta_*} \pi_{k+2}(S^k) \xrightarrow{i_{k+2*}} \pi^2_{k,2} \xrightarrow{p_{k+2*}} \pi_{k+2}(S^{k+1}),$$

i.e.

$$\to Z_2 \to Z_2 \to \pi^2_{k,2} \to Z_2 \to 0,$$

since p_{k+2*} is onto by (*b*). Also

$$i^{-1}_{k+2*}(0) = t_{k+2,2*}\, \pi_{k+2}(S^k),$$

which is generated by

$$h^*_{k,k+2}\{t_{k+2,2}\} = h^*_{k,k+2}\,2\{h_{k,k}\} = 2h^*_{k,k+2}\{h_{k,k}\} = 2\{h_{k+2,k}\} = 0.$$

Hence $i^{-1}_{k+2*}(0) = 0$; whence $\pi^2_{k,2}$ *has four elements.*

Note that thus Δ_* is trivial, and hence p_{k+3*} is 'onto'.

We now need to determine the structure of $\pi^2_{k,2}$. For this we look at the space P^{k+1}_{k-1} [see 2.3 (*d*)], which is of the same homotopy type as the Y^{k+1}_2 defined in 3.2 [see Appendix to part (II)]. M. G. Barratt showed in (1) that the homotopy classes of maps of P^{k+1}_{k-1} into itself can be turned into a group $(P^{k+1}_{k-1})^0(P^{k+1}_{k-1})$ when $k \geqslant 3$, and that this group is Z_4 [(1) 10.61]; and more particularly that, if α is the identical map of P^{k+1}_{k-1} onto itself, i' the identical map

$$\psi^{-1}_{k+3,3}\, i_{k+1,1} \colon S^k \to P^{k+1}_{k-1},$$

and p' the shrinking map

$$p_{k+2,1}\, \psi_{k+3,3} \colon P^{k+1}_{k-1} \to S^{k+1},$$

then α generates $(P^{k+1}_{k-1})^0(P^{k+1}_{k-1})$ and $2\alpha = \{i' h_{k,k+1} p'\}$.

Next consider the sequence

$$\to \pi_{k+2}(S^k) \xrightarrow{i'_*} \pi_{k+2}(P^{k+1}_{k-1}) \xrightarrow{p'_*} \pi_{k+2}(S^{k+1}) \to .$$

By virtue of the isomorphisms of Theorems 2.3 (*f*) and (*g*), this is exact and reduces to

$$(0) \to Z_2 \to \pi_{k+2}(P^{k+1}_{k-1}) \to Z_2 \to (0).$$

Let $\beta \colon S^{k+2} \to P^{k+1}_{k-1}$ be such that $p'^*\{\beta\} = \{h_{k+1,k+2}\}$. Then since

$$\{\beta\} \in \mathfrak{E}\pi_{k+2}(P^{k+1}_{k-1}) \quad \text{for } k \geqslant 5$$

by Corollary 3.2 (C) (k is odd), it induces a homomorphism

$$\beta^* \colon (P^{k+1}_{k-1})^0(P^{k+1}_{k-1}) \to \pi_{k+2}(P^{k+1}_{k-1}).$$

Thus, for $k \geqslant 5$,

$$\begin{aligned} 2\beta^*\alpha = \beta^*\, 2\alpha = \beta^*\{i' h_{k,k+1} p'\} &= \{i' h_{k,k+1} p' \beta\} \\ &= \{i' h_{k,k+1} h_{k+1,k+2}\} = \{i' h_{k,k+2}\} \neq 0 \end{aligned}$$

from (*c*) above. Thus, when $k \geqslant 5$, we have found an element, $\beta^*\alpha$, in $\pi_{k+2}(P^{k+1}_{k-1})$ twice which is non-zero. Hence $\pi_{k+2}(P^{k+1}_{k-1}) = Z_4$ for $k \geqslant 5$. For $k = 3$, the result follows from this and Corollary 3.2 (C). By Theorem 2.3 (*g*) we therefore have that

$$\pi^2_{k,2} = Z_4, \quad \textit{for odd } k \geqslant 3,$$

and is generated by any a such that $p_{k+2,1*}\, a = \{h_{k+1,k+2}\}$.

(*d*) When $p = 3$, (A) gives, when $k \geqslant 5$,

$$\xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+1}) \xrightarrow{\Delta_*} \pi_{k+3}(S^k) \xrightarrow{i_{k+3*}} \pi^3_{k,2} \xrightarrow{p_{k+3*}} \pi_{k+2}(S^{k+1}) \rightarrow$$

i.e.
$$\rightarrow Z_{24} \rightarrow Z_{24} \rightarrow \pi^3_{k,2} \rightarrow Z_2 \rightarrow 0$$

since p_{k+3*} is 'onto' by (*c*). Also

$$i^{-1}_{k+3*}(0) = t_{k+2,2*}\pi_{k+3}(S^k),$$

which is generated by

$$h^*_{k,k+3}\{t_{k+2,2}\} = h^*_{k,k+3}\,2\{h_{k,k}\} = 2\{h_{k,k+3}\}.$$

Hence $i^{-1}_{k+3*}(0) = 2Z_{24}$; *i.e.* $\pi^3_{k,2}$ *has four elements.*

Note that exactness implies that $\Delta_*\pi_{k+4}(S^{k+1}) = 2\pi_{k+3}(S^k)$, whence $\Delta^{-1}_*(0)$, and so the image of p_{k+4*}, is Z_2 generated by $12\{h_{k+1,k+4}\}$.

To determine the structure of $\pi^3_{k,2}$, we operate with $h^*_{k+2,k+3}$ on the section of the sequence given in 5.2 (*c*) and obtain the commutative diagram [cf. (3.2)]

$$\begin{array}{ccccccc}
\rightarrow \pi_{k+3}(S^k) & \xrightarrow{i_{k+3*}} & \pi^3_{k,2} & \xrightarrow{p_{k+2,1*}} & \pi_{k+3}(S^{k+1}) \rightarrow 0 \\
\uparrow h^* & & \uparrow h^* & & \uparrow h^* \\
\rightarrow \pi_{k+2}(S^k) & \xrightarrow{i_{k+2*}} & \pi^2_{k,2} & \xrightarrow{p_{k+2,1*}} & \pi_{k+2}(S^{k+1}) \rightarrow 0.
\end{array}$$

Now let a be a generator of $\pi^3_{k,2}$ such that

$$p_{k+2,1*}a = \{h_{k+1,k+3}\},$$

and a' a generator of $\pi^2_{k,2}$. Then

$$h^*p_{k+2,2*}a' = h^*\{h_{k+1,k+2}\} = \{h_{k+1,k+2}h_{k+2,k+3}\} = \{h_{k+1,k+3}\}$$

Hence
$$p_{k+2,1*}h^*a' = \{h_{k+1,k+3}\}.$$

Thus
$$h^*a' = a + i_{k+3*}b,$$

where $b \in \pi_{k+3}(S^k)$; and $2(h^*a') = 2a + 2i_{k+3}$.

But
$$2i_{k+3*}b = i_{k+3*}2b = 0,$$

from above; and further,

$$2(h^*a') = 2a'_*\{h_{k+2,k+3}\} = a'_*\,2\{h_{k+2,k+3}\} = 0.$$

Hence
$$2a = 0,$$

i.e.
$$\pi^3_{k,2} = Z_2 + Z_2 \textit{ for odd } k \geqslant 5,$$

and is generated by $\{i_{k+1,1}h_{k,k+3}\}$ and any a such that $p_{k+2,1*}a = \{h_{k+1,k+3}\}$.

(*e*) When $p = 3$, and $k = 3$, (A) gives

$$\xrightarrow{p_{7*}} \pi_7(S^4) \xrightarrow{\Delta_{7*}} \pi_6(S^3) \xrightarrow{i_{6*}} \pi^3_{3,2} \xrightarrow{p_{6*}} \pi_6(S^4) \rightarrow$$

i.e.
$$\rightarrow Z_\infty + Z_{12} \rightarrow Z_{12} \rightarrow \pi^3_{3,2} \rightarrow Z_2 \rightarrow 0$$

since p_{6*} is 'onto' by (*c*). Unfortunately in this case we cannot make use of Theorem 1.3 since $\pi_7(S^4) \neq \mathrm{E}\pi_6(S^3)$, but at any rate we have from Theorem 1.4 that

$$\Delta_{7*}\,\mathrm{E}\pi_6(S^3) = t_{5,2*}\,\pi_6(S^3),$$

which is generated by

$$h^*_{3,6}\{t_{5,2}\} = h^*_{3,6}\,2\{h_{3,3}\} = 2\{h_{3,6}\}$$

since $h^*_{3,6}$ is a homomorphism by Theorem IV of (**7**). Hence

$$\Delta_{7*}\,\mathrm{E}\pi_6(S^3) = 2\pi_6(S^3).$$

I have not been able to find a direct method of calculating $\Delta_{7*}\bar{p}_*\{h_{7,7}\}$. There is however another way of evaluating $\pi^3_{3,2}$. Consider the sequence associated with the fibring $V_{5,3}/S^2 \to V_{5,2}$:

i.e. $$\to \pi^4_{2,3} \to \pi^3_{3,2} \to \pi_5(S^2) \to \pi^3_{2,3} \to \pi^2_{3,2} \to \pi_4(S^2) \to.$$

I shall show in § 8.5 (*c*) and (*d*) respectively that $\pi^3_{2,3} = Z_2$ and $\pi^4_{2,3} = 0$. Hence the sequence is of the form

$$\to 0 \to \pi^3_{3,2} \to Z_2 \to Z_2 \to Z_4 \to Z_2 \to$$

since $\pi^2_{3,2} = Z_4$ by (*c*). Thus, by exactness, we have

$$\pi^3_{3,2} = Z_2,$$

generated by any a such that $p_{5,1*}a = \{h_{4,6}\}$. Note that i_{6*} is trivial and that this implies that $\Delta_{7*}\bar{p}_*\{h_{7,7}\} = \lambda\{h_{3,6}\}$, where λ is odd. Exactness then gives that the image of p_{7*} is the subgroup $Z_\infty + Z_2$ which is generated by $(2\bar{p}_*\{h_{7,7}\} - \lambda\mathrm{E}\{h_{3,6}\})$ and $6\mathrm{E}\{h_{3,6}\}$.

(*f*) When $p = 4$, and $k \geqslant 7$, (A) gives

$$\to \pi_{k+5}(S^{k+1}) \to \pi_{k+4}(S^k) \to \pi^4_{k,2} \xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+1}) \to.$$

But $\pi_{k+4}(S^k) = 0$, and by (*d*) the image of p_{k+4*} is Z_2. Hence

$$\pi^4_{k,2} = Z_2 \quad \textit{for odd } k \geqslant 7,$$

and is generated by $p^{-1}_{k+2,1*}\,12\{h_{k+1,k+4}\}$.

(*g*) When $p = 4$ and $k = 3$, (A) gives

$$\xrightarrow{p_{8*}} \pi_8(S^4) \xrightarrow{\Delta_{8*}} \pi_7(S^3) \xrightarrow{i_{7*}} \pi^4_{3,2} \xrightarrow{p_{7*}} \pi_7(S^4) \to$$

i.e. $$\to Z_2 + Z_2 \to Z_2 \to \pi^4_{3,2} \to Z_\infty + Z_2 \to 0$$

since the image of p_{7*} is $Z_\infty + Z_2$ by (*e*). Again we cannot make use of Theorem 1.3. But consider

$$i_{7*}\{h_{3,7}\} = i_{7*}\{h_{3,6}\,h_{6,7}\} = i_{7*}h^*_{6,7}\{h_{3,6}\} = h^*_{6,7}\,i_{6*}\{h_{3,6}\} = 0$$

since i_{6*} is trivial by (*e*). Hence $i_{7*}\,\pi_7(S^3) = 0$; and we have the result

$$\pi^4_{3,2} = Z_\infty + Z_2,$$

and is generated by

$$p_{5,1*}^{-1}(2\bar{p}_*\{h_{7,7}\}-\lambda\mathfrak{E}\{h_{3,6}\})\ (\lambda \text{ odd}) \quad \text{and} \quad p_{5,1*}^{-1}\,6\mathfrak{E}\{h_{3,6}\}.$$

Note that, by Theorem 1.4,

$$\Delta_{8*}\,\mathfrak{E}\pi_7(S^3) = t_{5,2*}\,\pi_7(S^3),$$

which is generated by

$$h_{3,7}^*\,2\{h_{3,3}\} = 2h_{3,7}^*\{h_{3,3}\},$$

$h_{3,7}^*$ being a homomorphism since [(7) Theorem IV] there is a multiplication on $S^3 = 2\{h_{3,7}\} = 0$. Thus $\Delta_{8*}\,\mathfrak{E}\pi_7(S^3) = 0$. But $i_{7*}\,\pi_7(S^3) = 0$, whence, by exactness,

$$\Delta_{8*}\,\bar{p}_*\{h_{7,8}\} = \{h_{3,7}\}.$$

So we have that the image of p_{8*} is the Z_2 subgroup $\mathfrak{E}\pi_7(S^3)$.

(*h*) When $p = 4$ and $k = 5$, (A) gives

$$\to \pi_{10}(S^6) \to \pi_9(S^5) \to \pi_{5,2}^4 \xrightarrow{p_{9*}} \pi_9(S^6) \to,$$

i.e.

$$\to 0 \to Z_2 \to \pi_{5,2}^4 \to Z_2 \subset Z_{24} \to 0$$

since by (*d*) the image of p_{9*} is Z_2. Thus $\pi_{5,2}^4$ *has four elements.* To determine its structure we operate with $h_{7,9}^*$ on that section of the sequence given in (*c*) to obtain the diagram

$$\begin{array}{ccccccc}
\to \pi_9(S^5) & \xrightarrow{i_{9*}} & \pi_{5,2}^4 & \xrightarrow{p_{7,1*}} & \pi_9(S^6) \to \\
\uparrow h^* & & \uparrow h^* & & \uparrow h^* \\
\to \pi_7(S^5) & \xrightarrow{i_{7*}} & \pi_{5,2}^2 & \xrightarrow{p_{7,1*}} & \pi_7(S^6) \to.
\end{array}$$

Let a be a generator of $\pi_{5,2}^4$ such that $p_{7,1*}a = 12\{h_{6,9}\}$, and let a' be a generator of $\pi_{5,2}^2$ $(= Z_4)$. Then

$$h^*p_{7,1*}a' = h^*\{h_{6,7}\} = \{h_{6,7}h_{7,9}\} = 12\{h_{6,9}\}.$$

Hence

$$p_{7,1*}h^*a' = 12\{h_{6,9}\}.$$

Thus $h^*a' = a+i_{9*}b$, where $b \in \pi_9(S^5)$; and $2h^*a' = 2a+2i_{9*}b$.

But

$$2i_{9*}b = i_{9*}\,2b = 0;$$

and further

$$2h^*a' = 2a'_*\{h\} = a'_*\,2\{h_{7,9}\} = 0.$$

Hence

$$2a = 0,$$

i.e.

$$\pi_{5,2}^4 = Z_2+Z_2,$$

and is generated by $\{i_{6,1}h_{5,9}\}$ and any a such that $p_{7,1*}a = 12\{h_{6,9}\}$.

(*i*) When $p = 5$ and $k \geqslant 7$, (A) gives

$$\to \pi_{k+6}(S^{k+1}) \to \pi_{k+5}(S^k) \to \pi_{k,2}^5 \to \pi_{k+5}(S^{k+1}) \to.$$

But $\pi_{k+5}(S^{k+1}) = 0 = \pi_{k+5}(S^k)$ when $k \geqslant 7$. Hence

$$\pi^5_{k,2} = 0 \quad \textit{for odd } k \geqslant 7.$$

(j) When $p = 5$ and $k = 3$, (A) gives

$$\xrightarrow{p_{9*}} \pi_9(S^4) \xrightarrow{\Delta_{9*}} \pi_8(S^3) \xrightarrow{i_{8*}} \pi^5_{3,2} \xrightarrow{p_{8*}} \pi_8(S^4) \to,$$

i.e.
$$\to Z_2 + Z_2 \to Z_2 \to \pi^5_{3,2} \to Z_2 \subset Z_2 + Z_2 \to 0$$

since by (g) the image of p_{8*} is Z_2. Again we cannot make use of Theorem 1.3; but as in (g) we consider

$$i_{8*}\{h_{3,8}\} = i_{8*}\{h_{3,6}h_{6,8}\} = i_{8*}h^*_{6,8}\{h_{3,6}\} = h^*_{6,8}i_{6*}\{h_{3,6}\} = 0$$

since i_{6*} is trivial by (e). Hence $i_{8*}\pi_8(S^3) = 0$, and we have the result

$$\pi^5_{3,2} = Z_2,$$

and is generated by $p^{-1}_{5,1*}\mathfrak{E}\{h_{3,7}\}$. Note that, by Theorem 1.4,

$$\Delta_{9*}\mathfrak{E}\pi_8(S^3) = t_{5,2*}\pi_8(S^3),$$

which is generated by

$$h^*_{3,8}2\{h_{3,3}\} = 2h^*_{3,8}\{h_{3,3}\} = 2\{h_{3,8}\} = 0,$$

$h^*_{3,8}$ being a homomorphism since there is a multiplication on S^3. Thus $\Delta_{9*}\mathfrak{E}\pi_8(S^3) = 0$. But $i_{8*}\pi_8(S^3) = 0$, whence, by exactness,

$$\Delta_{9*}\bar{p}_*\{h_{7,9}\} = \{h_{3,8}\}.$$

So the image of p_{9*} is the Z_2 subgroup generated by $\mathfrak{E}\{h_{3,8}\}$.

(k) When $p = 5$ and $k = 5$, (A) gives

$$\to \pi_{11}(S^6) \to \pi_{10}(S^5) \xrightarrow{i_{10*}} \pi^5_{5,2} \to \pi_{10}(S^6) \to,$$

i.e.
$$\to Z_\infty \to Z_2 \to \pi^5_{5,2} \to 0.$$

Since $\pi_{11}(S^6) \neq \mathfrak{E}\pi_{10}(S^5)$, we cannot use Theorem 1.3 to determine the kernel of i_{10*}, but have to use a special method. Note, however, that $\pi^5_{5,2}$ is at most Z_2.

Consider first the P^6_4 imbedded in $V_{7,2}$ [2.3 (d)], which is of the same homotopy type as Y^6_2 which consists of an S^5 to which a 6-cell has been attached by a map of degree two on its boundary. Then (Y^6_2, S^5) is a pair of the type considered in § 2 of (**12**), and so we have the exact sequence

$$\to \pi_{11}(S^6) \xrightarrow{H_\alpha} \pi_5(S^5) \xrightarrow{Q} \pi_{10}(Y^6_2, S^5) \to \pi_{10}(S^6) \to,$$

where H_α is defined as $\alpha_* \mathfrak{E}^{-6}H$ by

$$\pi_{11}(S^6) \xrightarrow{H} \pi_{11}(S^{11}) \xleftarrow{\mathfrak{E}^6} \pi_5(S^5) \xrightarrow{\alpha_*} \pi_5(S^5),$$

H being the Hopf invariant and α the attaching map, of degree 2. Since there exist maps : $S^{11} \to S^6$ of Hopf invariant 2 (**11**), but not of Hopf invariant 1 (**20**), and, since α is of degree 2, the image of Q is Z_4. But $\pi_{10}(S^6) = 0$. Hence

$$\pi_{10}(P_4^6, S^5) \approx \pi_{10}(Y_4^6, S^5) = Z_4.$$

Now $V_{7,2}$, being a fibre space over S^6 with fibre S^5, is an eleven-dimensional space, and the cellular decomposition of (**22**) shows that, besides P_4^6, $V_{7,2}$ contains just one other cell, an E^{11}. Thus

$$\pi_{11}(V_{7,2}, P_4^6) = Z_\infty, \qquad \pi_{10}(V_{7,2}, P_4^6) = 0,$$

and the homotopy sequence of the triple $(V_{7,2}, P_4^6, S^5)$:

$$\to \pi_{11}(P_4^6, S^5) \xrightarrow{\psi_{8,3*}} \pi_{11}(V_{7,2}, S^5) \to \pi_{11}(V_{7,2}, P_4^6) \xrightarrow{\delta''_*} \pi_{10}(P_4^6, S^5) \to \pi_{10}(V_{7,2}, S^5) \to$$

becomes $\quad \to \pi_{11}(P_4^6, S^5) \to Z_\infty \to Z_\infty \to Z_4 \to 0 \to.$

Hence $\quad \psi_{8,3*}\pi_{11}(P_4^6, S^5) = 0,$

and δ''_*: $\pi_{11}(V_{7,2}, P_4^6)$ is 'onto'.

Next consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \uparrow & & & & \\
 & & \to \pi_{11}(S^6) & \xrightarrow{\Delta_*} & \pi_{10}(S^5) & \xrightarrow{i_*} & \pi_{10}(V_{7,2}) & \to \pi_{10}(S^6) \to & \\
 & & \uparrow{\scriptstyle p'_*\psi_{8,3*}} & & \uparrow{\scriptstyle \psi_{7,2*}} & & \uparrow{\scriptstyle \psi_{8,3*}} & \uparrow{\scriptstyle p'_*\psi_{8,3*}} & \\
\to \pi_{11}(P_4^6, S^5) & \xrightarrow{\delta'_*} & \pi_{10}(S^5) & \xrightarrow{i'_*} & \pi_{10}(P_4^6) & \xrightarrow{j'_*} & \pi_{10}(P_4^6, S^5) \to & & \\
 & & & & \uparrow{\scriptstyle \delta_*} & \nearrow{\scriptstyle \delta_*} & & & \\
 & & & & \pi_{11}(V_{7,2}, P_4^6), & & & & \\
 & & & & \uparrow & & & &
\end{array}$$

in which both the horizontal and vertical sequences are exact. Since δ''_* is 'onto', so is j'_*. Since $\psi_{8,3*}$ is trivial and $\psi_{7,2*}$ is an isomorphism,

$$\delta'_*\pi_{11}(P_4^6, S^5) = 0.$$

Hence $i'^{-1}_*(0) = 0$, and thus $\pi_{10}(P_4^6)$ is an extension of Z_2 by Z_4. I show below that $i'_*\{h_{5,10}\}$ can be halved in $\pi_{10}(P_4^6)$, whence $\pi_{10}(P_4^6) = Z_8$. But $j'_*\delta_* = \delta''_*$, which is 'onto'. Hence δ_* is 'onto', and

$$\pi_{10}(V_{7,2}) = 0.$$

To prove that $i'_*\{h_{5,10}\}$ can be halved in $\pi_{10}(P_4^6)$, consider Y_2^6 and let $\langle Y_2^6$ and $Y_2^6\rangle$ be two cones with different vertices based on Y_2^6, so that

$\langle Y_2^6 \cap Y_2^6 \rangle = Y_2^6$ and $\langle Y_2^6 \cup Y_2^6 \rangle = \mathfrak{E}Y_2^6 \simeq Y_2^7$.† Then consider the diagram

$$\begin{array}{ccc} \to \pi_{12}(Y_2^7; \langle Y_2^6, Y_2^6 \rangle) & \xrightarrow{\delta_+} & \pi_{11}(\langle Y_2^6, Y_2^6 \rangle) \to \\ & & \downarrow \delta_{1*} \\ & \pi_{10}(S^5) \xrightarrow{i'_*} & \pi_{10}(Y_2^6), \end{array}$$

where δ_+, δ_{1*}, and i'_* are the triad-boundary, homotopy-boundary, and injection homomorphisms. Since $\langle Y_2^6$ is contractible, δ_{1*} is an isomorphism. Let δ_{2*} be the corresponding isomorphism on $\pi_r(Y_2^6\rangle, Y_2^6)$, and let $h'_{5,5}$ and $h'_{5,6}$ be generators of $\pi_6(Y_2^6\rangle, Y_2^6)$ and $\pi_7(\langle Y_2^6, Y_2^6)$. Then

$$\begin{aligned} \delta_{1*}\delta_+[h'_{5,5}, h'_{5,6}] &= -\delta_{1*}[\delta_{2*}h'_{5,5}, h'_{5,6}], \quad \text{by 4.3 of (4)}, \\ &= -[\delta_{2*}h'_{5,5}, \delta_{1*}h'_{5,6}], \quad \text{by 3.5 of (4)}. \end{aligned}$$

But $$\delta_{2*}h'_{5,5} = i''_*\{h_{5,5}\}, \qquad \delta_{1*}h'_{5,6} = i''_*\{h_{5,6}\}.$$

Hence $$\delta_{1*}\delta_+[h'_{5,5}, h'_{5,6}] = i''_*\{h_{5,10}\}.$$

If we now shrink Y_2^6 to a point in $(Y_2^7; \langle Y_2^6, Y_2^6\rangle)$, we obtain a triad $(X; A, B)$, where A and B are two different copies of Y_2^7 having only a single point in common. By §§ 5, 6, and especially p. 403 of (**2**), we have that

$$\pi_{12}(Y_2^7; \langle Y_2^6, Y_2^6\rangle) \approx \pi_{12}(X; A, B) = Z_4;$$

and further that, if α is a generator, $[h'_{5,5}, h'_{5,6}] = 2\alpha$. Hence

$$i''_*\{h_{5,10}\} = 2\delta_{1*}\delta_+\alpha,$$

and so can be halved in $\pi_{10}(Y_2^6)$.

5.3. $k = 1$. As in § 5.2, we have from § 2.3 (*b*) that $\{t_{3,2}\} = 2\{h_{1,1}\}$. Then

(*a*) when $p = 0$, (A) gives

$$\xrightarrow{p_{2*}} \pi_2(S^2) \xrightarrow{\Delta_*} \pi_1(S^1) \xrightarrow{i_{1*}} \pi^0_{1,2} \to \pi_1(S^2),$$

i.e. $$\to Z_\infty \to Z_\infty \to \pi^0_{1,2} \to 0,$$

where $i_{1*}^{-1}(0)$ is generated by $\{t_{3,2}\}$, i.e. by $2\{h_{1,1}\}$.

Thus $\pi^0_{1,2} = Z_2$, *generated by* $\{i_{2,1}h_{1,1}\}$.

Note that $\Delta_*^{-1}(0) = 0$, and thus p_{2*} is trivial.

(*b*) When $p \geqslant 1$, (A) gives

$$\to \pi_{p+1}(S^1) \xrightarrow{i_{p+1*}} \pi^p_{1,2} \xrightarrow{p_{p+1*}} \pi_{p+1}(S^2) \xrightarrow{\Delta_*} \pi_p(S^1) \to.$$

But $\pi_{p+1}(S^1) = 0$ when $p \geqslant 1$; and we also have from (*a*) that p_{2*} is trivial.

Thus $\pi^1_{1,2} = 0$; and $\pi^p_{1,2} \approx \pi_{p+1}(S^2)$ *for* $p \geqslant 2$. The values of $\pi^p_{1,2}$ for $p \geqslant 2$ are then as shown in the tables.

† I am much indebted to Dr. M. G. Barratt for the following argument.

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CONNEXIONS FOR SYSTEMS OF PARALLEL DISTRIBUTIONS

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1. Introduction

THIS paper may be regarded as a sequel to a previous paper by A. G. Walker† (**2**).

In (**2**) Walker proved the theorem:

THEOREM 1. *For any system of distributions over a manifold‡ there exists in the large an affine connexion with respect to which the distributions are parallel and which is symmetric if the system is integrable.*

I gave a short proof of this existence theorem in (**5**); here I give another proof by a method introduced in (**4**), which, like Walker's, has the advantage over the method of (**5**) in that an explicit expression for a suitable connexion is given. I then show that the connexions obtained by Walker in (**2**) are precisely those obtained here when all arbitrary parameters are set equal to zero.

In general there is considerable freedom of choice of a connexion which makes a system of distributions parallel, for this depends on the independent choice of an arbitrary symmetric connexion and an arbitrary complementary distribution. However, a definite procedure is described which makes correspond to a given complementary distribution a connexion giving parallelism whose torsion is independent of the choice of the arbitrary symmetric connexion. In particular a complete system of distributions has a system of torsion tensors canonically associated with it whose vanishing is a necessary and sufficient condition for the system to be integrable.

Finally I prove that an n-dimensional manifold which admits n linearly independent vector fields $(\mathbf{e}_i)$ such that $[\mathbf{e}_i, \mathbf{e}_j] = \mathbf{0}$, can be given a Euclidean metric with respect to which the vector fields are parallel. If, in addition, the manifold is compact, I show that it is homeomorphic to a torus. This contains as an immediate corollary the result proved

† I wish to thank Professor Walker for showing me his paper in manuscript.

‡ Unless otherwise stated, differentiable manifolds and distributions are of class C^∞.

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by Ishihara and Obata (**1**) that, under our hypothesis, the manifold has its pth Betti number equal to $\binom{n}{p}$.

2. Asymmetric connexions

If an n-dimensional manifold M admits a distribution D of m-planes, then it necessarily admits a complementary distribution D_0 of l-planes where $l+m=n$, and the structure group $L(n)$ of the principal fibre bundle of the tangent bundle of M is reducible to the product $L(m)\times L(l)$. Similarly, if M admits a system of disjoint distributions

$$\{D_\rho\} \quad (\rho = 1, 2, \ldots, p),$$

then there exists a complementary distribution D_0 of l-planes where $l+\sum_{\rho=1}^{p} m_\rho = n$, and $m_\rho = \dim D_\rho$. In this case the structure group is reducible to

$$L(m_1)\times L(m_2)\times \ldots \times L(m_p)\times L(l).$$

I shall use suffixes i, j, k for the index range R, $(1,\ldots, n)$; α, β, γ for the range R_1, $(1,\ldots, m_1)$; λ, μ, ν for the range R_2, $(m_1+1,\ldots, m_1+m_2)$; ξ, η, ζ for the range R_3, $(m_1+m_2+1,\ldots, m_1+m_2+m_3)$; etc.; r, s, t for the range R_0, $(n-l+1,\ldots, n)$.

Attention may then be restricted to a set of distinguished frames

$$(\mathbf{e}_i) = (\mathbf{e}_\alpha, \mathbf{e}_\lambda, \mathbf{e}_\xi, \ldots, \mathbf{e}_r),$$

where, at each point P, $(\mathbf{e}_\alpha)$ spans the plane of D_1, $(\mathbf{e}_\lambda)$ spans the plane of D_2, $(\mathbf{e}_\xi)$ spans the plane of D_3, etc., and $(\mathbf{e}_r)$ spans the plane of the complementary distribution D_0. Moreover, the linear transformations which relate any two distinguished frames at P must belong to the restricted group $L(m_1)\times L(m_2)\times \ldots \times L(m_p)\times L(l)$.

It is convenient to introduce the following notation. If q is any suffix of the range R_σ, then q' takes values over all other ranges. If q is any suffix of the range R_σ $(\sigma \neq 0)$, then $\tilde{q}$ takes values over all other ranges, excluding R_0.

Let $(\omega^i) = (\omega^\alpha, \omega^\lambda, \omega^\xi, \ldots, \omega^r)$ be a set of Pfaffian forms forming the dual basis to that given by the distinguished basis $(\mathbf{e}_i) = (\mathbf{e}_\alpha, \mathbf{e}_\lambda, \mathbf{e}_\xi, \ldots, \mathbf{e}_r)$. Then locally D_1 is given by the equations $\omega^{\alpha'} = 0$, D_2 by $\omega^{\lambda'} = 0$, D_3 by $\omega^{\xi'} = 0$, etc., and D_0 by $\omega^{r'} = 0$.

Let $(\omega^i{}_j)$ be the components with respect to the basis (ω^i) of an affine connexion, defined over some neighbourhood of M. Then the absolute differentials of the basis vectors $(\mathbf{e}_j)$ are given by

$$D\mathbf{e}_j = \omega^i{}_j \mathbf{e}_i. \tag{2.1}$$

It follows that a necessary and sufficient condition for the given system of distributions $\{D_\rho\}$ to be parallel is

$$\omega^{\kappa'}{}_{\kappa} = 0, \qquad \kappa \in R_\rho \quad (\rho = 1, 2, \ldots, p). \tag{2.2}$$

Let $(\Omega^i{}_j)$ be the components with respect to (ω^i) of any symmetric connexion which is defined globally over M: the existence of such a connexion is implied by the existence of a positive definite Riemannian metric which can always be defined globally over M. Then the symbols $(t^i{}_j)$ defined by the equation

$$\omega^i{}_j = \Omega^i{}_j - t^i{}_j \tag{2.3}$$

are the components of a tensorial differential form of order one. Conversely, if $(t^i{}_j)$ are components of such a tensorial form, then $(\omega^i{}_j)$ defined by (2.3) are components of an affine connexion. Moreover, if $(t^i{}_j)$ is defined globally over M, so is the connexion $(\omega^i{}_j)$.

From (2.2), (2.3) it follows that a necessary and sufficient condition on $(t^i{}_j)$ for parallelism of the distributions $\{D_\rho\}$ is

$$t^{\kappa'}{}_{\kappa} = \Omega^{\kappa'}{}_{\kappa}, \qquad \kappa \in R_\rho \quad (\rho = 1, 2, \ldots, p). \tag{2.4}$$

The remaining components of $(t^i{}_j)$ are unrestricted, and may conveniently be taken as zero. It is easily verified that the set $(t^i{}_j)$ so defined transforms as a tensorial form under the restricted transformation-group. There is thus defined globally over M an affine connexion with respect to which the distributions $\{D_\rho\}$ are parallel. However this connexion will not generally be symmetric. We consider next conditions for the existence of a *symmetric* connexion with respect to which the system of distributions shall be parallel.

3. Symmetric connexions

It is well known (cf. **3**) that on a manifold with a symmetric affine connexion every parallel distribution is necessarily integrable. It follows that in order that the system $\{D_\rho\}$ shall be parallel with respect to a symmetric connexion, it is necessary that each distribution D_ρ shall be integrable. However, this condition is not sufficient because, if D_ρ, D_σ are two disjoint distributions, each being parallel with respect to a symmetric connexion, then the sum $D_\rho + D_\sigma$ is necessarily parallel, but the sum of two integrable distributions is not necessarily integrable. Following the terminology of Walker (**2**) I shall say that a *system* $\{D_\rho\}$ *is integrable* if every sum of distributions is integrable.

Write

$$D = \sum_{\rho=1}^{p} D_\rho, \qquad D^\sigma = \sum_{\rho \neq \sigma} D_\rho. \tag{3.1}$$

Then, as Walker pointed out in (2), the system $\{D_\rho\}$ is integrable if and only if each of the set D, D^1, D^2,..., D^p is separately integrable. This follows from the fact that every sum of distributions of the system $\{D_\rho\}$ is the intersection of distributions of the system $\{D, D^\sigma\}$, and the result that the intersection of integrable distributions is integrable.

We observe that D is given by $\omega^r = 0$; D^1 by $\omega^r = 0$, $\omega^\alpha = 0$; D^2 by $\omega^r = 0$, $\omega^\lambda = 0$; D^3 by $\omega^r = 0$, $\omega^\xi = 0$, etc. The conditions of integrability of the system $\{D_\rho\}$ thus reduce to

$$\left.\begin{array}{ll} d\omega^r = 0 & \text{when } \omega^\rho = 0,\ \rho \in R_0 \\ d\omega^\alpha = 0 & \text{when } \omega^\rho = 0,\ \rho \in R_0 + R_1 \\ d\omega^\lambda = 0 & \text{when } \omega^\rho = 0,\ \rho \in R_0 + R_2 \\ d\omega^\xi = 0 & \text{when } \omega^\rho = 0,\ \rho \in R_0 + R_3 \\ \cdots & \end{array}\right\}. \tag{3.2}$$

A necessary and sufficient condition for the connexion $(\omega^i{}_j)$ to be symmetric (torsion-free) is

$$d\omega^i + \omega^i{}_j \wedge \omega^j = 0. \tag{3.3}$$

Since the connexion $(\Omega^i{}_j)$ is symmetric, we have

$$d\omega^i + \Omega^i{}_j \wedge \omega^j = 0. \tag{3.4}$$

From (2.3), (3.3), (3.4) it follows that, if $(\omega^i{}_j)$ is symmetric, then

$$t^i{}_j \wedge \omega^j = 0. \tag{3.5}$$

Conversely, when (3.5) is satisfied, the connexion defined by (2.3) will be symmetric. Hence our problem is reduced to defining $(t^i{}_j)$ globally over M so that (2.4) is satisfied and so that (3.5) is a consequence of (3.2).

Write

$$t^i{}_j = c^i{}_{jk}\omega^k, \qquad \Omega^i{}_j = \gamma^i{}_{jk}\omega^k, \tag{3.6}$$

where the coefficients $c^i{}_{jk}$ are to be determined.

The conditions of integrability (3.2) of the system of distributions together with relations (3.4) which express the symmetry of $(\Omega^i{}_j)$ now give

$$\gamma^r_{s't'} = \gamma^r_{t's'}; \qquad \gamma^\alpha_{\beta\bar{\gamma}} = \gamma^\alpha_{\bar{\gamma}\beta}, \qquad \gamma^\lambda_{\bar{\mu}\bar{\nu}} = \gamma^\lambda_{\bar{\nu}\bar{\mu}}, \quad \text{etc.} \tag{3.7}$$

The conditions for parallelism (2.4) lead to the relations

$$c^{\alpha'}_{\beta i} = \gamma^{\alpha'}_{\beta i}, \qquad c^{\lambda'}_{\mu i} = \gamma^{\lambda'}_{\mu i}, \quad \text{etc.},$$

which can be written in the form

$$c^r_{s'i} = \gamma^r_{s'i}; \qquad c^\alpha_{\beta i} = \gamma^\alpha_{\beta i}, \qquad c^\lambda_{\bar{\mu} i} = \gamma^\lambda_{\bar{\mu} i}, \quad \text{etc.} \tag{3.8}$$

The conditions (3.5) which must be satisfied if the required connexion is symmetric become

$$c^i_{jk} = c^i_{kj}. \tag{3.9}$$

These equations are compatible with (3.8) because of relations (3.7).

Choose coefficients $(c^i{}_{jk})$ so that conditions (3.8) are satisfied. Certain coefficients will not be determined by equations (3.8), (3.9), namely

$$c^r_{st};\ c^\alpha_{\beta\gamma},\ c^\alpha_{t\beta},\ c^\alpha_{\gamma s},\ c^\alpha_{st};\ c^\lambda_{\mu\nu},\ c^\lambda_{t\mu},\ c^\lambda_{\nu s},\ c^\lambda_{st};\ \text{etc.}$$

These are arbitrary except for symmetry relations among themselves, i.e.

$$c^r_{st} = c^r_{ts};\ c^\alpha_{\beta\gamma} = c^\alpha_{\gamma\beta},\ c^\alpha_{t\beta} = c^\alpha_{\beta t},\ c^\alpha_{st} = c^\alpha_{ts};\ \text{etc.} \tag{3.10}$$

We take all these coefficients to be zero, and show that the matrix $(t^i{}_j)$ is now uniquely determined.

We have $\quad c^\alpha_{\tilde\gamma i} = \gamma^\alpha_{\tilde\gamma i}, \qquad c^\alpha_{\gamma\tilde\beta} = \gamma^\alpha_{\tilde\beta\gamma}, \qquad c^\alpha_{t\tilde\beta} = \gamma^\alpha_{\tilde\beta t},$

together with $\quad c^\alpha_{\beta\gamma} = c^\alpha_{\beta t} = c^\alpha_{s\gamma} = c^\alpha_{st} = 0.$

Hence $\quad t^\alpha{}_\gamma = \gamma^\alpha_{\tilde\beta\gamma}\omega^{\tilde\beta}, \qquad t^\alpha{}_t = \gamma^\alpha_{\tilde\beta t}\omega^{\tilde\beta}.$

Also, from (2.4), we have $\quad t^\alpha{}_{\tilde\beta} = \Omega^\alpha{}_{\tilde\beta},$

so that the first row of the partitioned matrix $(t^i{}_j)$ is determined. In the same way the second, third,..., pth rows of the matrix are determined.

In order to obtain the last row, we use the relations

$$c^r_{t'i} = \gamma^r_{t'i}, \qquad c^r_{ts'} = \gamma^r_{s't}, \qquad c^r_{st} = 0,$$

from which $\quad t^r{}_t = \gamma^r_{s't}\omega^{s'}.$

Also, from (2.4), we have $\quad t^r{}_{t'} = \Omega^r{}_{t'},$

and hence the last row is determined.

It is easily verified that the set $(t^i{}_j)$ defined above transforms as a tensorial form under the restricted transformation-group, so that the corresponding connexion given by (2.3) is defined globally over the manifold, makes the distributions parallel, and is of zero torsion when the system is integrable. This completes the proof of Theorem 1.

4. Transformation to a natural frame

In this section I show that under a transformation from a distinguished frame to a frame naturally associated with the coordinate system, the connexion obtained in the previous section with all arbitrary parameters set equal to zero gives precisely the connexion previously obtained by Walker (2).

Consider the natural basis $(\mathbf{e}^*_i)$ defined by

$$\mathbf{e}^*_i = \partial/\partial x^i,$$

with the dual basis $\quad \omega^{*i} = dx^i.$

The transformations connecting the distinguished frames $(\mathbf{e}_i)$, (ω^i) with these natural frames are

$$\mathbf{e}_a = \lambda^j{}_a \mathbf{e}^*{}_j, \qquad \omega^a = \mu^a{}_j \omega^{*j},$$

where $$\lambda^j{}_a \mu^a{}_i = \delta^j{}_i.$$

Under this transformation the components of the tensorial form $t^a{}_b$ satisfy

$$t^{*i}{}_j = \lambda^i{}_a t^a{}_b \mu^b{}_j. \tag{4.1}$$

For reasons of space I omit details of the calculation of $(t^{*i}{}_j)$, where $(t^a{}_b)$ is given in the previous section. In the course of the calculation it is convenient to introduce symbols as follows.

Write $$\underset{\rho}{a}{}^i{}_j = \lambda^i{}_q \mu^q{}_j \quad \text{where } q \text{ runs over } R_\rho,$$

$$a^i{}_j = \lambda^i{}_{p'} \mu^{p'}{}_j \quad \text{where } p \text{ runs over } R_0,$$

$$\underset{\rho}{b}{}^i{}_j = a^i{}_j - \underset{\rho}{a}{}^i{}_j;$$

write

$$\Omega^{*i}{}_j = \Gamma^i_{jk}\, dx^k, \qquad \omega^{*i}{}_j = L^i_{jk}\, dx^k, \qquad t^{*i}{}_j = -T^{*i}_{jk}\, \omega^{*k}.$$

Define $T^i_{jk}(\mathbf{a}, \Gamma)$ by the relation

$$T^i_{jk}(\mathbf{a}, \Gamma) = -a^i{}_{u,j} a^u{}_k - a^i{}_{u,k} a^u{}_j + a^i{}_{u,v} a^u{}_k a^v{}_j$$

where a comma denotes covariant differentiation with respect to Γ^i_{jk}. Define $T^i_{jk}\left(\underset{\rho}{\mathbf{b}}, \Gamma\right)$ by the same formula with $\mathbf{a}$ replaced by $\underset{\rho}{\mathbf{b}}$. We find that the elements in the ρth row of the matrix $(t^a{}_b)$ together contribute the term $a^i{}_u T^u_{jb}\left(\underset{\rho}{\mathbf{b}}, \Gamma\right)$ in the expression for T^{*i}_{jk}, while the elements in the last row together yield the term $T^i_{jk}(\mathbf{a}, \Gamma)$. We finally get

$$L^i_{jk} = \Gamma^i_{jk} + T^i_{jk}(\mathbf{a}, \Gamma) + a^i{}_u \sum_{\rho=1}^{p} T^u_{jk}\left(\underset{\rho}{\mathbf{b}}, \Gamma\right), \tag{4.2}$$

which is precisely the formula obtained by Walker (**2**, (17)).

5. The connexion given by (4.2) depends upon the choice of the symmetric connexion Γ^i_{jk} and the arbitrary complementary distribution D_0. However, I show that once D_0 has been chosen, the torsion tensor defined by

$$t^i_{jk} = \tfrac{1}{2}(L^i_{jk} - L^i_{kj}) \tag{5.1}$$

is *independent of the particular symmetric connexion chosen.*

Define tensors σ^i_{jk}, $\underset{\rho}{\sigma}{}^i_{jk}$ by the relations

$$\sigma^i_{jk} = \tfrac{1}{2}\{T^i_{jk}(\mathbf{a}, \Gamma) - T^i_{kj}(\mathbf{a}, \Gamma)\}, \tag{5.2}$$

$$\underset{\rho}{\sigma}{}^i_{jk} = \tfrac{1}{2} a^i{}_u \left\{T^u_{jk}\left(\underset{\rho}{\mathbf{b}}, \Gamma\right) - T^u_{kj}\left(\underset{\rho}{\mathbf{b}}, \Gamma\right)\right\}, \tag{5.3}$$

so that
$$t^i_{jk} = \sigma^i_{jk} + \sum_{\rho=1}^{p} \sigma^i_{\rho jk}. \tag{5.4}$$

Then, after some calculation, we find that

$$\sigma^i_{jk} = \tfrac{1}{2} a^u{}_k a^v{}_j \left(\frac{\partial a^i{}_u}{\partial x^v} - \frac{\partial a^i{}_v}{\partial x^u} \right), \tag{5.5}$$

$$\sigma^i_{\rho jk} = \tfrac{1}{2} b^u{}_{\rho k} b^v{}_{\rho j} \left(\frac{\partial a^i{}_v}{\partial x^u} - \frac{\partial a^i{}_u}{\partial x^v} \right), \tag{5.6}$$

showing that each of the tensors σ^i_{jk}, $\sigma^i_{\rho jk}$ is independent of Γ, and hence t^i_{jk} is independent of Γ.

These results may be put in another form involving the bracket operation of vector fields. Denote by P the linear operator corresponding to the tensor $a^i{}_j$ which maps the module $\mathscr{M}$ of vector fields over $\mathscr{M}$ into itself. Similarly let P_ρ denote the linear operator corresponding to the tensor $b^i{}_{\rho j}$, and let I be the identity operator. Let σ be the bilinear mapping $\sigma\colon \mathscr{M} \times \mathscr{M} \to \mathscr{M}$ defined by

$$[\sigma(\boldsymbol{\lambda}, \boldsymbol{\mu})]^i = \sigma^i_{jk} \lambda^j \mu^k. \tag{5.7}$$

Similarly define σ_ρ by the relation

$$\left[\sigma_\rho(\boldsymbol{\lambda}, \boldsymbol{\mu})\right]^i = \sigma^i_{\rho jk} \lambda^j \mu^k. \tag{5.8}$$

Then a simple calculation shows that

$$(I - P)[P\boldsymbol{\lambda}, P\boldsymbol{\mu}] = 2\sigma(\boldsymbol{\lambda}, \boldsymbol{\mu}), \tag{5.9}$$

and
$$\left(P - P_\rho\right)\left[P\boldsymbol{\lambda}, P_\rho \boldsymbol{\mu}\right] = 2\sigma_\rho(\boldsymbol{\lambda}, \boldsymbol{\mu}), \tag{5.10}$$

giving confirmation that the tensors σ^i_{jk}, $\sigma^i_{\rho jk}$ are independent of the symmetric connexion Γ.

If the given system of distributions is *complete* (i.e. at each point the planes of the distributions span the whole tangent space), then the choice of D_0 does not arise and there is thus associated with the system a set of tensors σ^i_{jk}, $\sigma^i_{\rho jk}$ whose vanishing is a necessary and sufficient condition for the system of distributions to be integrable.

6. A system of n linearly independent vector fields $(\mathbf{e}_i)$ such that $[\mathbf{e}_i, \mathbf{e}_j] = \mathbf{0}$ is said to form a *commutative n-frame*. We now prove

THEOREM 2. *A compact differentiable manifold of class C^r $(r \geqslant 3)$ which admits a commutative n-frame is homeomorphic to a torus.*

The conditions $[\mathbf{e}_i, \mathbf{e}_j] = \mathbf{0}$ are necessary in order that the vector

fields $(\mathbf{e}_i)$ shall be parallel with respect to a symmetric connexion. We now prove that when these conditions are satisfied, the vector fields are parallel with respect to a *locally Euclidean metric*. Certainly the existence of the frame $(\mathbf{e}_i)$ implies the existence of a dual frame (ω^i) of n linearly independent 1-forms which gives rise to the Riemannian metric

$$ds^2 = \sum_{i=1}^{n} (\omega^i)^2.$$

A short calculation now shows that the conditions $[\mathbf{e}_i, \mathbf{e}_j] = \mathbf{0}$ imply $d\omega^i = 0$. Hence the Riemannian metric is flat. It follows that, when M admits a commutative n-frame, it can be metrized globally by a flat metric with respect to which the vector fields are parallel. Hence a *simply-connected* manifold which admits a commutative n-frame is necessarily homeomorphic with Euclidean n-space.

Consider now the simply connected manifold $\tilde{M}$ which covers M, equipped with the locally Euclidean metric induced from M. Then $\tilde{M}$ is Euclidean n-space. Since the n vectors $(\mathbf{e}_i)$ are globally parallel over M, the homogeneous holonomy group of M consists of just the identity while the full holonomy group is a discrete group generated by translations parallel to the vectors $(\mathbf{e}_i)$. Since M is compact, it follows that M is obtained from $\tilde{M}$ as a fundamental region bounded by hyperplanes whose directions are determined by the vectors $(\mathbf{e}_i)$. From this it follows that M is an n-dimensional torus, and the theorem is proved.

As a corollary it follows that the pth Betti number of M is $\binom{n}{p}$, which is the result obtained by Ishihara and Obata (**1**).

In conclusion, I should like to thank the referee for pointing out that some of the results of §§ 2–5 include as a special case results already obtained by A. Fröhlicher ['Zur Differentialgeometrie der komplexen Strukturen', *Math. Annalen* 120 (1955), 50–95]. It is hoped to deal with this matter elsewhere.

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THE INTERSECTION OF THE KÖTHE-TOEPLITZ MAXIMAL MATRIX RINGS

By H. S. ALLEN (*London*)

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LET α_i $(i \in J)$ be the set of all perfect sequence spaces. The ring $\sum(\alpha_i)$ of all matrices which map α_i into α_i is a maximal Köthe–Toeplitz ring [(3), 315]. In this paper the class of matrices which belong to the intersection of all the rings $\sum(\alpha_i)$ $(i \in J)$ will be identified.

The set of all matrices which belong to $\sum(\alpha_i)$ and which have a two-sided reciprocal in $\sum(\alpha_i)$ is a group with respect to matrix multiplication, and I shall denote this group by $G(\alpha_i)$ [(1), 112]. The intersection $G(\alpha_i) \cap G(\alpha_j)$ $(i \neq j)$, is a non-empty proper subset of $G(\alpha_i)$ and of $G(\alpha_j)$, and an example is known in which $G(\alpha_i) \cap G(\alpha_j)$ is not a group [(4), 105]. In the case $\alpha_i \supset \alpha_j$, $G(\alpha_i) \cap G(\alpha_j)$ is a proper subgroup of $G(\alpha_i)$ and of $G(\alpha_j)$ [(1), 119]. It is known that the intersection $g = \bigcap_{i \in J} G(\alpha_i)$ is a group [(1), 119], and in this paper the elements of g will be identified.

I shall denote the set of all matrices which are row-bounded and column-bounded by $\mathfrak{B}$, and the set of all diagonal matrices by $\mathfrak{D}$.

The intersection of all the maximal rings of matrices which transform a perfect convergence-free space [(2), 280] into itself is determined first.

THEOREM 1. *A matrix A maps every perfect convergence-free space into itself if and only if $A = B+D$, where $B \in \mathfrak{B}$ and $D \in \mathfrak{D}$.*

Suppose that $A = \{a(n,k)\}$ maps every perfect convergence-free space into itself. The spaces ϕ and σ are convergence-free, and hence $A \in \sum(\phi) \cap \sum(\sigma)$; and it follows that A is row-finite and column-finite [(3), 301]. I prove first that the number of non-zero elements of A above the principal diagonal is finite. Suppose that A has infinitely many non-zero elements above the principal diagonal and let l_n $(n = 1, 2,...)$ be the least non-negative integer such that $a(n,k) = 0$ $(k > l_n)$. Then the set S of integers n such that $l_n > n$ is an infinite set, and there are integers n_j $(j = 1, 2,...)$ in S such that $n_j < l_{n_j} < n_{j+1}$. We determine a sequence $u = \{u(k)\}$ such that $u(l_{n_j}) \neq 0$ $(j = 1, 2,...)$, $u(k) = 0$ when k is not in the sequence $\{l_{n_j}\}$, and whose A-transform has non-zero co-ordinates with suffixes n_j $(j = 1, 2,...)$. Let $u(l_{n_1}) = 1$, and determine $u(l_{n_2}) \neq 0$ such that

$$a(n_2, l_{n_1})u(l_{n_1}) + a(n_2, l_{n_2})u(l_{n_2}) \neq 0.$$

Quart. J. Math. Oxford (2), 7 (1956), 277-9.

When $u(l_{n_j}) \neq 0$ has been determined, we choose $u(l_{n_{j+1}}) \neq 0$ such that

$$\sum_{i=1}^{j+1} a(n_{j+1}, l_{n_i}) u(l_{n_i}) \neq 0.$$

Let M be the set of all positive integers which are not in the sequence $\{l_{n_j}\}$; then

$$n_j \in M \quad (j = 1, 2, \ldots).$$

Let β be the space of all sequences x such that the number of non-zero coordinates of x with suffixes in M is finite. It follows from the definitions that β is convergence-free and that M is an F-set for β [(**2**), 281]. Hence β contains all sequences which have only a finite number of non-zero coordinates with suffixes in every F-set for β, and it follows that β is perfect [(**2**), 282]. Hence $A\beta \subseteq \beta$. We have $u \in \beta$ since $u(k) = 0$ $(k \in M)$, and $(Au)_{n_j} \neq 0$ $(j = 1, 2, \ldots)$. The contradiction proves that A has only a finite number of non-zero elements above the principal diagonal. If λ is any perfect convergence-free space, then λ^* is perfect and convergence-free [(**2**), 280], and hence $A \in \sum (\lambda^*)$ and $A' \in \sum (\lambda)$ [(**3**), 300]. It follows that the number of non-zero elements of A below the principal diagonal is finite. This proves the theorem.

COROLLARY. *If a K_r-matrix (K_c-matrix or Hilbert matrix) A maps every perfect convergence-free space into itself, then A maps every normal space which contains ϕ into itself* [(**2**), 278].

We have $A = B+D$, where $B \in \mathfrak{B}$ and $D \in \mathfrak{D}$. In this case the set of diagonal elements of D is bounded, and the result follows.

THEOREM 2. *The ring $\bigcap_{i \in J} \sum (\alpha_i)$ is the ring of all matrices $B+D$, where $B \in \mathfrak{B}$ and $D \in \mathfrak{D}$, and the set of diagonal elements of D is bounded.*

The ring $\bigcap_{i \in J} \sum (\alpha_i)$ is a sub-ring of $\sum (\sigma_\infty)$, the ring of K_r-matrices [(**3**), 298], and hence every matrix which belongs to this sub-ring can be expressed in this form.

Conversely, every perfect space contains ϕ and is normal [(**2**), 278], and hence every matrix of this class belongs to the ring $\bigcap_{i \in J} \sum (\alpha_i)$.

THEOREM 3. *A matrix A belongs to the group g if and only if there is a non-negative integer n and positive constants m and M such that $A = B+D$, where $B = (b_{i,k}) \in \mathfrak{B}$, $D = (d_i) \in \mathfrak{D}$, and*

(i) $d_i = 0$ $(1 \leqslant i \leqslant n)$, $m \leqslant |d_i| \leqslant M$ $(i > n)$,

(ii) $b_{i,k} = 0$ *if either i or k exceeds n,*

(iii) *the $n \times n$ matrix $B_1 = (b_{i,k}^{(1)})$ with elements $b_{i,k}^{(1)} = b_{i,k}$ $(i, k = 1, 2, \ldots, n)$ is non-singular.*

Suppose that $A \in g$; then A has only a finite number of non-zero elements outside the principal diagonal. Let n be the greatest of the row and column suffixes of the non-zero elements of A which are not in the principal diagonal. Then A can be expressed in the form

$$A = B+D,$$

where $B = (b_{i,k})$ satisfies (ii), $D = (d_i) \in \mathfrak{D}$, and D is chosen so that $d_i = 0, 1 \leqslant i \leqslant n$. Then $|d_i| \leqslant M$ for $i > n$, by Theorem 2. The matrix A has a two-sided reciprocal in the group g. The unique two-sided reciprocal of $B+D$ can be expressed in the form B_1+D_1, where (*a*) $B_1 \in \mathfrak{B}$, (*b*) the greatest of the row and column suffixes of the non-zero elements of B_1 is n, (*c*) $D_1 = (d_i^{(1)}) \in \mathfrak{D}$, $d_i^{(1)} = 0\ (1 \leqslant i \leqslant n)$, $d_i^{(1)} d_i = 1\ (i > n)$. Since the set of diagonal elements of D_1 is bounded, it follows that the conditions are necessary.

If the conditions are satisfied, then A belongs to the ring $\bigcap_{i \in J} \sum (\alpha_i)$, and has a two-sided reciprocal in this ring, and hence $A \in g$.

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SOME FURTHER WORK ON L^2-SOLUTIONS OF THE WAVE EQUATION

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1. An L^2-solution, $\phi = \phi(x, y)$, of the partial differential equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\{\lambda-q(x,y)\}\phi = 0, \tag{1.1}$$

where $q(x, y)$ is a real continuous function and x, y vary over the whole plane, has continuous second-order partial derivatives and satisfies the inequality

$$0 < \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\phi(x,y)|^2\, dxdy < \infty. \tag{1.2}$$

In (**1**) it was shown that, if (1.1) has an L^2-solution for one complex value of the parameter λ, then it has an L^2-solution for every complex value of λ, and the Green's function $G(x, y, \xi, \eta, \lambda)$ corresponding to (1.1) and the whole plane is not unique. The L^2-solutions of (1.1) for real values of λ were examined only in the case when (1.1) had no L^2-solution for a complex value of λ, and, hence, no L^2-solution for any complex value of λ (Green's function is unique). The result is that, in this case, (1.1) has an L^2-solution if and only if the function

$$H(x, y, \xi, \eta, u) = \frac{1}{\pi}\lim_{v\to 0}\int_0^u \operatorname{im} G(x, y, \xi, \eta, u'+iv)\, du', \tag{1.3}$$

when regarded as a function of u, is discontinuous at $u = \lambda$. It has been shown by Titchmarsh (**2**) that this function exists and is of bounded variation in any finite interval. Thus, its points of discontinuity are enumerable and they may be said to form the *discrete spectrum* of (1.1) and the whole plane.

When Green's function is not unique, points of discontinuity of an H-function depend, in general, on the particular Green's function taken in (1.3), and hence they cannot be expressed solely in terms of a property of (1.1) and the whole plane. In this case it is no longer justifiable to speak of a *unique* spectrum belonging to (1.1) and the whole plane. The object of this paper is to show that, in the case when Green's function is not unique, (1.1) has an L^2-solution for those real values of λ which

form either points of discontinuity of an H-function, or points lying inside an open interval throughout which H is constant.

2. Let μ (real) be a point of discontinuity of $H = H(u)$ and

$$\phi = \phi(x, y, \xi, \eta)$$

the corresponding saltus, so that

$$\phi(x, y, \xi, \eta) = H(x, y, \xi, \eta, \mu+0) - H(x, y, \xi, \eta, \mu-0) \tag{2.1}$$

is not identically zero, and, as in (**1**),

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{\phi(x, y, \xi, \eta)\}^2 \, dxdy < \infty. \tag{2.2}$$

The method of proof used in (**1**) [217–19] to show that

$$\phi(x, y) = \phi(x, y, \xi, \eta)$$

satisfies (1.1) with $\lambda = \mu$ is valid only if Green's function is unique. Here I give a more general method of proof also valid in the case when Green's function is not unique.

THEOREM 1. *Let μ (real) be a point of discontinuity of an H-function. Then, whether Green's function is unique or not,* (1.1) *with $\lambda = \mu$ has an L^2-solution.*

The converse of the theorem holds only if Green's function is unique. It will be shown that (2.1) supplies the required solution; the theorem then follows by (2.2).

Let C be the square whose centre is at μ and whose sides are parallel to the λ-axes and of length $2l$. Let λ be complex and lie inside C. Then, by (6.2) of (**1**),

$$G(x, y, \xi, \eta, \lambda) = \frac{1}{2\pi i} \int_C \frac{G(x, y, \xi, \eta, z)}{z-\lambda} \, dz + \int_{u=\mu-l}^{u=\mu+l} \frac{dH(x, y, \xi, \eta, u)}{u-\lambda}, \tag{2.3}$$

where the first integral on the right-hand side is a contour integral taken round the perimeter of C and the second integral is a Stieltjes integral taken over $\mu-l \leqslant u \leqslant \mu+l$. The argument used to prove this formula is valid whether Green's function is unique or not. Also, by the definition (1.3) of an H-function,

$$0 = \frac{1}{2\pi i} \int_C G(x, y, \xi, \eta, z) \, dz + \int_{u=\mu-l}^{u=\mu+l} dH(x, y, \xi, \eta, u). \tag{2.4}$$

The following formula is also required [see, for example, (11.7) of (**2**)]:

$$G(x_0, y_0, \xi, \eta, \lambda) = \frac{1}{2\pi} \iint_{r \leqslant R} \left\{ \frac{1}{2}\left(1 - \frac{r^2}{R^2}\right) - \log \frac{R}{r} \right\} \{q(x,y) - \lambda\} G(x, y, \xi, \eta, \lambda) r \, dr d\theta + \\ + \frac{1}{\pi R^2} \iint_{r \leqslant R} G(x, y, \xi, \eta, \lambda) r \, dr d\theta, \quad (2.5)$$

where $r = \sqrt{\{(x-x_0)^2 + (y-y_0)^2\}}$, $R < \sqrt{\{(\xi - x_0)^2 + (\eta - y_0)^2\}}$, and λ is any complex number.

Write z in place of λ in (2.5), divide the resulting equation by $z - \lambda$, and integrate round C. Thus

$$\int_C \frac{G(x_0, y_0, \xi, \eta, z)}{z - \lambda} \, dz \\ = \frac{1}{2\pi} \iint_{r \leqslant R} \left\{ \frac{1}{2}\left(1 - \frac{r^2}{R^2}\right) - \log \frac{R}{r} \right\} \{q(x,y) - \lambda\} \left\{ \int_C \frac{G(x, y, \xi, \eta, z)}{z - \lambda} \, dz \right\} r \, dr d\theta - \\ - \frac{1}{2\pi} \iint_{r \leqslant R} \left\{ \frac{1}{2}\left(1 - \frac{r^2}{R^2}\right) - \log \frac{R}{r} \right\} \left\{ \int_C G(x, y, \xi, \eta, z) \, dz \right\} r \, dr d\theta + \\ + \frac{1}{\pi R^2} \iint_{r \leqslant R} \left\{ \int_C \frac{G(x, y, \xi, \eta, z)}{z - \lambda} \, dz \right\} r \, dr d\theta. \quad (2.6)$$

For 'almost all' contours C, each term on the right-hand side of this equation exists as an absolutely convergent integral, and the inversions are permissible by (12.4) of (**2**).

Divide both sides of (2.6) by $2\pi i$ and subtract the resulting equation from (2.5). Then, by (2.3) and (2.4),

$$\int_{u=\mu-l}^{u=\mu+l} \frac{dH(x_0, y_0, \xi, \eta, u)}{u - \lambda} \\ = \frac{1}{2\pi} \iint_{r \leqslant R} \left\{ \frac{1}{2}\left(1 - \frac{r^2}{R^2}\right) - \log \frac{R}{r} \right\} \{q(x,y) - \lambda\} \times \\ \times \left\{ \int_{u=\mu-l}^{u=\mu+l} \frac{dH(x, y, \xi, \eta, u)}{u - \lambda} \right\} r \, dr d\theta - \\ - \frac{1}{2\pi} \iint_{r \leqslant R} \left\{ \frac{1}{2}\left(1 - \frac{r^2}{R^2}\right) - \log \frac{R}{r} \right\} \left\{ \int_{u=\mu-l}^{u=\mu+l} dH(x, y, \xi, \eta, u) \right\} r \, dr d\theta + \\ + \frac{1}{\pi R^2} \iint_{r \leqslant R} \left\{ \int_{u=\mu-l}^{u=\mu+l} \frac{dH(x, y, \xi, \eta, u)}{u - \lambda} \right\} r \, dr d\theta.$$

Making $l \to 0$ in this equation, we see that the saltus ϕ may be written in place of G and μ in place of λ in (2.5). By a well-known method [see, for example, 11 (i) of (**2**)] it can then be shown that ϕ satisfies (1.1) with $\lambda = \mu$.

3. Let λ and λ' be two complex numbers and let $h = h(x, y)$ be any function of L^2. Consider the integral equation

$$f(x, y) = h(x, y) + (\lambda' - \lambda)\Phi(x, y, \lambda', f), \tag{3.1}$$

where $$\Phi(x, y, \lambda', f) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x, y, \xi, \eta, \lambda') f(\xi, \eta)\, d\xi d\eta.$$

By the well-known method of successive approximation it can be shown that (3.1) has an L^2-solution $f = f(x, y)$ provided that

$$|\lambda - \lambda'| < |\operatorname{im} \lambda'|.$$

Thus, let $f_1 = h$ and

$$f_{n+1} = h + (\lambda' - \lambda)\Phi(x, y, \lambda', f_n) \quad (n = 1, 2, \ldots). \tag{3.2}$$

From the integral inequality for the Φ-function, [see (14.2) of (**2**)], each f_n is L^2 and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f_{n+1} - f_n|^2\, dxdy \leqslant \left|\frac{\lambda' - \lambda}{\operatorname{im} \lambda'}\right|^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f_n - f_{n-1}|^2\, dxdy.$$

Iterating this inequality, we see that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f_{n+1} - f_n|^2\, dxdy \leqslant A^2 k^{2n},$$

where A is independent of n and

$$k = \left|\frac{\lambda' - \lambda}{\operatorname{im} \lambda'}\right| < 1.$$

Thus, by Minkowski's inequality,

$$\begin{aligned}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f_m - f_n|^2\, dxdy\right)^{\frac{1}{2}} &\leqslant \sum_{r=n}^{m-1} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f_{r+1} - f_r|^2\, dxdy\right)^{\frac{1}{2}} \\ &\leqslant A \sum_{r=n}^{m-1} k^r \\ &\to 0,\end{aligned}$$

as m and n tend to infinity. Thus, as $n \to \infty$, f_n tends in mean square to a function f of L^2. By (3.2), f_n also tends to a limit which may be taken to be equal to the mean-square limit everywhere; (3.1) follows by making $n \to \infty$ in (3.2).

Suppose that h is an L^2-solution of (1.1) when $\lambda = \lambda'$. Then

$$\begin{aligned}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\lambda-q(x,y)\right)f &= \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\lambda-q(x,y)\right)h+\\ &\quad+(\lambda'-\lambda)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\lambda-q(x,y)\right)\Phi(\lambda',f)\\ &= (\lambda'-\lambda)\{-h+f+(\lambda-\lambda')\Phi(\lambda',f)\}\\ &= 0,\end{aligned}$$

by (3.1) and the differential equation satisfied by the Φ-function [see (14.3) of (**2**)]. The result shows that, if (1.1) has an L^2-solution when $\lambda = \lambda'$, then it has an L^2-solution for any λ lying inside the circle with centre λ' and radius $|\operatorname{im}\lambda'|$. By proceeding from one circle to another, we obtain yet another proof of the invariance property for the non-uniqueness of Green's function. The difference between this method and the two earlier proofs given in (**1**) is that here we proceed by actually constructing L^2-solutions.

4. Having shown the existence of a solution of (3.1) we can now prove the following result:

THEOREM 2. *Let Green's function be not unique. Let μ (real) be a point lying inside an open interval throughout which $H = H(u)$ is constant. Then* (1.1) *with $\lambda = \mu$ has an L^2-solution.*

Let λ' be a complex number tending to μ. The argument used in § 6 of (**1**) holds also for the case when Green's function is not unique. Hence, by proceeding along lines similar to those used in § 7 of the same paper, we deduce that $\Phi(x,y,\lambda',f)$ may be continued analytically across the real λ'-axis in the neighbourhood of μ. Also, as $\lambda' \to \mu$,

$$\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\Phi(x,y,\lambda',f)|^2\,dxdy\right)^{\frac{1}{2}} \leqslant \kappa\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f|^2\,dxdy\right)^{\frac{1}{2}}, \qquad (4.1)$$

where κ is a fixed number independent of λ' and f but depending on μ.

Let $\lambda = \mu+i\nu$ and $\lambda' = \mu+i/2\kappa$, where $0 < \nu < 1/2\kappa$. Let h be an L^2-solution of (1.1) with $\lambda = \lambda'$, and f be an L^2-solution of (3.1). By (3.1) and (4.1),

$$\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f|^2\,dxdy\right)^{\frac{1}{2}} \leqslant \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|h|^2\,dxdy\right)^{\frac{1}{2}}+\frac{1}{2}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f|^2\,dxdy\right)^{\frac{1}{2}},$$

i.e.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f|^2\,dxdy \leqslant 4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|h|^2\,dxdy.$$

Hence, as $\nu \to 0$, f is weakly compact and has a weak limit, say ϕ, as ν approaches 0 through a certain sequence of values. By (3.1), f tends to a limit in the ordinary sense. This ordinary limit must be equal to the weak limit almost everywhere and, since ϕ has so far only been defined almost everywhere, may be taken equal to ϕ everywhere. Making $\nu \to 0$ in (3.1), we thus get

$$\phi = h + (\lambda' - \mu)\Phi(x, y, \lambda', \phi);$$

and, by proceeding as in the last section, it follows that ϕ is the required L^2-solution.

5. If $q(x, y)$ is a function of $r = \sqrt{(x^2+y^2)}$ only, then there is a more precise classification of the L^2-solutions of (1.1) for real values of λ.

THEOREM 3. *Let $q(x, y) = q(r)$, where $r = \sqrt{(x^2+y^2)}$, give rise to a Green's function which is not unique; so that, for every complex value of λ, the partial differential equation*

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \{\lambda - q(r)\}\phi = 0 \tag{5.1}$$

has an L^2-solution $\phi = \phi(x, y)$. Then (5.1) has an L^2-solution for every real value of λ.

The converse of the theorem follows from results given in (1).

Let

$$\phi_n(r) = \int_0^{2\pi} \phi(r\cos\theta, r\sin\theta)e^{in\theta}\, d\theta \quad (n = 0, \pm 1, \pm 2, \ldots).$$

Then

$$\int_0^\infty r|\phi_n(r)|^2\, dr \leqslant 2\pi \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\phi(x, y)|^2\, dxdy < \infty,$$

and every $\phi_n(r)$ is not identically zero. Thus

$$\omega_n(r) = r^{\frac{1}{2}}\phi_n(r)$$

is $L^2(0, \infty)$, and (5.1) reduces to the set of ordinary differential equations

$$\frac{d^2\omega_n(r)}{dr^2} + \left\{\lambda - q(r) - \frac{n^2 - \frac{1}{4}}{r^2}\right\}\omega_n(r) = 0 \quad (n = 0, \pm 1, \pm 2, \ldots). \tag{5.2}$$

Reversing the procedure, if $\omega_n(r)$ is an $L^2(0, \infty)$ solution of (5.2), we have

$$\phi(x, y) = r^{-\frac{1}{2}}\omega_n(r)e^{in\theta}$$

as an L^2-solution of (5.1).

It is easily seen that (5.2) has one, and only one, solution which is

$O(r^{\frac{1}{2}})$, as $r \to 0$. The theorem will therefore follow if $q(r)$ gives rise to a Green's function which is not unique for the problem given by

$$\frac{d^2\psi(r)}{dr^2} + \{\lambda - q(r)\}\psi(r) = 0 \quad (0 \leqslant r < \infty), \tag{5.3}$$

i.e. if $q(r)$ gives Weyl's limit-circle case at $r = \infty$. For, if this be the case, every solution of (5.3) is L^2 at $r = \infty$ (λ real or complex), and, by a well-known result [see references (**1**) and (**9**) quoted in (**1**)], the same result holds for (5.2).

On the contrary suppose that $q(r)$ gives Weyl's limit-point case at $r = \infty$; i.e. Green's function is unique for $q(r)$ and a one-dimensional interval $0 < a \leqslant r < \infty$. Then, whenever Green's function is unique, we have a corresponding Green's formula [see Theorem 2 in (**1**)] given in this case by

$$\int_a^\infty \left[f(r)\left\{q(r) - \frac{d^2}{dr^2}\right\}\overline{g(r)} - \overline{g(r)}\left\{q(r) - \frac{d^2}{dr^2}\right\}f(r)\right] dr = f(a)\overline{g'(a)} - f'(a)\overline{g(a)},$$

where f and $\{q(r) - d^2/dr^2\}f$, etc., are all $L^2(a, \infty)$. Now for a given complex λ there is at least one integer $n = n(\lambda)$ such that $\omega_n(r) \not\equiv 0$. Putting $f = g = \omega_n$, we obtain, on using (5.2),

$$\begin{aligned}\int_a^\infty \left[\omega_n(r)\left\{\bar{\lambda} - \frac{n^2 - \frac{1}{4}}{r^2}\right\}\overline{\omega_n(r)} - \overline{\omega_n(r)}\left\{\lambda - \frac{n^2 - \frac{1}{4}}{r^2}\right\}\omega_n(r)\right] dr \\ = \omega_n(a)\overline{\omega_n'(a)} - \omega_n'(a)\overline{\omega_n(a)} \\ = a\{\phi_n(a)\overline{\phi_n'(a)} - \phi_n'(a)\overline{\phi_n(a)}\}.\end{aligned}$$

When $a \to 0$, it follows that

$$(\bar{\lambda} - \lambda)\int_0^\infty |\omega_n(r)|^2\, dr = 0,$$

which is a contradiction.

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SINGULAR SYSTEMS OF TWO DIFFERENTIAL EQUATIONS

By D. H. PARSONS (*Reading*)

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A SYSTEM of n partial differential equations of the first order, with two independent variables x and y, and n dependent variables, is said to be *singular* if the characteristic determinant is identically zero for all values of dy/dx.† In such a system, every curve in the (x, y)-plane is in a sense a characteristic, and the usual existence theorem cannot be applied: for the dependent variables cannot be specified arbitrarily along any curve. Such systems occur in differential geometry, and in applied mathematics; but hitherto, no general existence theorem has been established. Indeed, particular examples admit general integrals of widely differing types, so that the difficulty of treating singular systems generally is considerable. In the present paper, I shall deal with the simplest case, that of singular systems of two equations only. I shall show that every singular system of two partial differential equations of the first order, with two independent and two dependent variables, is necessarily a linear system. I shall also investigate the integrals of such a system.

Consider the system

$$F_1(x, y, z_1, z_2, p_1, p_2, q_1, q_2) = 0, \qquad F_2(x, y, z_1, z_2, p_1, p_2, q_1, q_2) = 0 \quad (1)$$

($p_i = \partial z_i/\partial x$, $q_i = \partial z_i/\partial y$; $i = 1, 2$) where, by the hypothesis that the system is singular,

$$\begin{vmatrix} \dfrac{\partial F_1}{\partial p_1}dy - \dfrac{\partial F_1}{\partial q_1}dx & \dfrac{\partial F_1}{\partial p_2}dy - \dfrac{\partial F_1}{\partial q_2}dx \\ \dfrac{\partial F_2}{\partial p_1}dy - \dfrac{\partial F_2}{\partial q_1}dx & \dfrac{\partial F_2}{\partial p_2}dy - \dfrac{\partial F_2}{\partial q_2}dx \end{vmatrix} \equiv 0, \quad (2)$$

for all values of the ratio dy/dx, and for any set of values of the variables which satisfies the two equations. I shall suppose that $F_1 = 0$ and $F_2 = 0$ are distinct analytic equations in the neighbourhood of a set of initial values which satisfies the equations: and we may further suppose the equations written in fully reduced form, so that they may be solved for

† E. Goursat, *Leçons sur l'Intégration des Équations aux Derivées Partielles du Second Ordre*, 2, Ch. X, 322 (Paris, 1898).

two of their arguments. This being so, it follows that any Jacobians which are zero for any set of values of the variables satisfying the system (1) are zero identically, and not as a consequence of equations (1).

Now from (2) we see that

$$\frac{\partial(F_1, F_2)}{\partial(p_1, p_2)} = \frac{\partial(F_1, F_2)}{\partial(q_1, q_2)} \equiv 0.$$

It follows that from the system (1) we may deduce an equivalent analytic system, of the form

$$h(x, y, z_1, z_2, p_1, p_2) = 0, \qquad k(x, y, z_1, z_2, q_1, q_2) = 0. \tag{3}$$

We may suppose that not both of $\partial h/\partial p_1$, $\partial h/\partial p_2$, and not both of $\partial k/\partial q_1$, $\partial k/\partial q_2$ are identically zero; for otherwise we should have an equation of the form

$$\chi(x, y, z_1, z_2) = 0.$$

Then either we have a relation between x and y, which is impossible, or else we could solve for z_2, say, in terms of z_1, x, y. The problem would then be reduced, by substituting this value in the other equation, to that of a single equation with one dependent variable, a case which we discard.

Suppose, therefore, that the first equation contains p_1, so that, for general values of the variables satisfying the equation, $\partial h/\partial p_1 \neq 0$. We can thus solve for p_1, so that this equation can be written

$$p_1 + P(x, y, z_1, z_2, p_2) = 0.$$

Then the second equation must contain q_1. For suppose, if possible, that $\partial k/\partial q_1 \equiv 0$, but $\partial k/\partial q_2 \not\equiv 0$. Then, solved for q_2, the equation would be of the form

$$q_2 + Q_2(x, y, z_1, z_2) = 0,$$

and the determinant corresponding to that on the left of (2) is

$$\begin{vmatrix} dy & \dfrac{\partial P}{\partial p_2} dy \\ 0 & -dx \end{vmatrix} = -dx dy \not\equiv 0.$$

The system is then non-singular, contrary to hypothesis. We therefore have $\partial k/\partial q_1 \not\equiv 0$; and, choosing values of the variables for which $\partial k/\partial q_1 \neq 0$, we can solve the second equation for q_1, and the system becomes

$$p_1 + P(x, y, z_1, z_2, p_2) = 0, \qquad q_1 + Q(x, y, z_1, z_2, q_2) = 0.$$

The condition corresponding to (2) is now

$$\begin{vmatrix} dy & \dfrac{\partial P}{\partial p_2}dy \\ -dx & -\dfrac{\partial Q}{\partial q_2}dx \end{vmatrix} = \left(\frac{\partial P}{\partial p_2}-\frac{\partial Q}{\partial q_2}\right)dxdy \equiv 0,$$

and we therefore have $$\frac{\partial P}{\partial p_2}=\frac{\partial Q}{\partial q_2}.$$

But P does not contain q_2, and Q does not contain p_2; therefore

$$\frac{\partial P}{\partial p_2}=\frac{\partial Q}{\partial q_2}=B_1(x,y,z_1,z_2),$$

and, on integration,

$$P=B_1p_2+C_1, \qquad Q=B_1q_2+D_1,$$

B_1, C_1, D_1 being functions of x, y, z_1, z_2. The system of equations is thus the linear system,

$$p_1+B_1p_2+C_1=0, \qquad q_1+B_1q_2+D_1=0.$$

If it should happen that $\partial h/\partial p_1 \equiv 0$, but $\partial h/\partial p_2 \not\equiv 0$, we repeat the reasoning with p_1 and p_2, q_1 and q_2 permuted, thereby showing that $\partial k/\partial q_2 \not\equiv 0$. The condition (2) now becomes

$$-\begin{vmatrix} 0 & \dfrac{\partial h}{\partial p_2}dy \\ -\dfrac{\partial k}{\partial q_1}dx & -\dfrac{\partial k}{\partial q_2}dx \end{vmatrix} = \frac{\partial h}{\partial p_2}\frac{\partial k}{\partial q_1}dydx \equiv 0,$$

and therefore $\partial k/\partial q_1 \equiv 0$. The equations then do not contain p_1 or q_1, and can be solved for p_2 and q_2.

We therefore see that, in every case, the given system is of the form

$$Ap_1+Bp_2+C=0, \qquad Aq_1+Bq_2+D=0, \tag{4}$$

where A, B, C, D are functions of x, y, z_1, z_2, and A, B are not both identically zero: and, conversely, we see at once that every system of this form satisfies the condition (2). This is the first of the required results.

To investigate the integrals of the system (4), we first observe, multiplying the first by dx, the second by dy, and adding, that every integral of (4) satisfies the equation

$$A\,dz_1+B\,dz_2+C\,dx+D\,dy=0; \tag{5}$$

and, conversely, every integral of (5) which does not involve any relationship between x and y is an integral of the system (4). There

are now several possibilities. The equation (5), an equation of Pfaff in four variables, x, y, z_1, z_2, can be of class 3 or of class 1. If it be of class 1, we can, by a change of variables, reduce it to the form

$$du = 0, \tag{6}$$

where u is a function of x, y, z_1, z_2. The general integral of (6) is

$$u = C, \tag{7}$$

where C is an arbitrary constant. Since A, B are not both identically zero, it follows that $\partial u/\partial z_1$, $\partial u/\partial z_2$ are not both identically zero, say $\partial u/\partial z_1 \not\equiv 0$. Then, choosing a value of C such that, for a set of values of the variables satisfying the equation $u = C$, $\partial u/\partial z_1 \neq 0$, we may take z_2 to be an arbitrary function of x and y and solve (7) for z_1. The general integral of the system (4) then contains one arbitrary function (z_2) and one arbitrary constant.

If, on the other hand, (5) is of class 3 (which is the general case), we can make a change of variables, which reduces (5) to the canonical form

$$du - v\,dw = 0, \tag{8}$$

in which the fourth variable does not appear. The general two-dimensional integral of (8) is

$$u = f(w), \qquad v = f'(w), \tag{9}$$

where f is an arbitrary function of w. The equation (8) also admits the two-dimensional integral

$$u = a, \qquad w = b, \tag{10}$$

where a and b are arbitrary constants.

Now, if the equations (9), or the equations (10), can be solved for z_1 and z_2, we have an integral of the system (4), the integral (9) containing an arbitrary function and its derivative, and the integral (10) two arbitrary constants. But, if we can eliminate z_1 and z_2 either between the equations (9) or the equations (10), obtaining a relationship between x and y, the system (9), or (10) as the case may be, though giving an integral of (5), does not give an integral of (4) in the proper sense. We can solve (9) for z_1 and z_2 if

$$\frac{\partial\{u - f(w), v - f'(w)\}}{\partial(z_1, z_2)} \neq 0,$$

i.e.

$$\frac{\partial(u, v)}{\partial(z_1, z_2)} + f'(w)\frac{\partial(v, w)}{\partial(z_1, z_2)} + f''(w)\frac{\partial(w, u)}{\partial(z_1, z_2)} \neq 0.$$

We can also solve (10) for z_1 and z_2 if

$$\frac{\partial(w,u)}{\partial(z_1,z_2)} \neq 0.$$

It follows therefore (except perhaps for exceptional values of the variables) that, if any two of u, v, w are distinct functions of z_1 and z_2 when x and y are regarded as constants, then, for general forms of the function f, we can solve the equations (9) for z_1 and z_2, and the given system (4) admits a general integral, containing an arbitrary function of one variable and its derivative, this integral being represented by (9) in which we express u, v, w in terms of x, y, z_1, z_2.

If, in addition, w and u are distinct functions of z_1 and z_2, so that, for general values of the variables, $\partial(w,u)/\partial(z_1,z_2) \neq 0$, the given system (4) also admits the integral represented by (10), containing two arbitrary constants, which is not included in the general integral.

But, if $$\frac{\partial(v,w)}{\partial(z_1,z_2)} = \frac{\partial(w,u)}{\partial(z_1,z_2)} = \frac{\partial(u,v)}{\partial(z_1,z_2)} \equiv 0,$$

then for all forms of $f(w)$, the integral (9) involves a relationship between x and y, and the same is true of (10). The given system (4) then admits no integrals of this kind. For example, the system

$$p_1+p_2 = y, \qquad q_1+q_2 = 0,$$

which is obviously inconsistent, leads to an equation of Pfaff

$$d(z_1+z_2)-y\,dx = 0$$

whose integrals are

$$z_1+z_2 = f(x), \quad y = f'(x), \quad \text{and} \quad x = a, \quad z_1+z_2 = b.$$

It should be noticed that the given system may also admit certain exceptional integrals which are not included in the general integral (9), or in the integral (10): for the equation (5) may admit singular integrals, which are not obtained by the reduction to canonical form†, and one or more of these integrals may provide an integral of the system (4). These integrals are, however, found in advance by algebraic calculations only. Again, it is evident that, if the four equations $A = B = C = D = 0$ be consistent and admit a common solution for z_1 and z_2 in terms of x and y, we once more have a singular integral of the given system (4).

† E. Goursat, *Leçons sur le Problème de Pfaff*, Ch. IV, § 47, 186–90 (Paris, 1922).

Example

$$\frac{\partial z_1}{\partial x} - xy(z_1^2+z_2^2)\frac{\partial z_2}{\partial x} - 2x\{1-y^2(z_1^2+z_2^2)\} = 0,$$

$$\frac{\partial z_1}{\partial y} - xy(z_1^2+z_2^2)\frac{\partial z_2}{\partial y} - 2y\{1-x^2(z_1^2+z_2^2)\} = 0.$$

Here the equation corresponding to (5) is

$$dz_1 - xy(z_1^2+z_2^2)\,dz_2 - 2x\{1-y^2(z_1^2+z_2^2)\}\,dx - 2y\{1-x^2(z_1^2+z_2^2)\}\,dy = 0,$$

which we write in the canonical form

$$d\{z_1-(x^2+y^2)\} - xy(z_1^2+z_2^2)\,d\{z_2-2xy\} = 0.$$

The general integral is represented by the equations

$$z_1-(x^2+y^2) = f(z_2-2xy), \qquad xy(z_1^2+z_2^2) = f'(z_2-2xy),$$

which, for general forms of the arbitrary function f, can be solved for z_1 and z_2. We also have the integral

$$z_1 = a+x^2+y^2, \qquad z_2 = b+2xy,$$

where a and b are arbitrary constants.

A NOTE ON BORSUK'S ANTIPODAL-POINT THEOREM

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1. Introduction

BORSUK in his paper 'Drei Sätze über die n-dimensionale euklidische Sphäre' (2) proves the theorem:

THEOREM 1. *Let $p \to p^*$ be the continuous antipodal involution $z \to -z$ on the n-sphere S^n; then every map $f\colon S^n \to S^n$ which is antipodal, i.e. $f(p^*) = [f(p)]^*$, has odd degree: in other words, the induced homology map*

$$f_*\colon H_n(S^n; Z_2) \to H_n(S^n; Z_2),$$

where Z_2 is the group of integers (mod 2), *is an epimorphism, and hence an isomorphism.*

Jaworowski (7) has generalized this by proving the theorem:

THEOREM 2. *Let M be a compact metric space such that $H_i(M; Z_2) = 0$ $(i < n)$ and ϕ a continuous involution on M without fixed points. Then every mapping $f\colon M \to S^n$ such that $f\{\phi(x)\} \neq f(x)$ for all $x \in M$ induces an epimorphism*

$$f_*\colon H_n(M; Z_2) \to H_n(S^n; Z_2).$$

I prove Jaworowski's theorem for a more general space M and mention some similar theorems for maps ϕ of order m.

2. Main theorem

We denote by $\bar{H}_n(M; K)$ the group $H_0\{K(G, 1); H_n(M; K)\}$, the 0th homology group of a given group G with the local coefficient group $H_n(M; K)$. The meaning of this group will be explained later.

All the results mentioned above are particular cases of the following general theorem:

Quart. J. Math. Oxford (2), 7 (1956), 293-300.

THEOREM 3. *Let L, M be arcwise connected Hausdorff spaces. Let G be a finite abelian group which operates, without fixed points, on L and M: that is to say, for each $g \in G$, there are homeomorphisms*

$$g: X \to X \quad (X = L, M)$$

such that $gx \neq x$ if $g \neq 1$ $(x \in X)$. Moreover let $H_i(X, K) = 0$ for $i < n$ and let K be an abelian coefficient group. Let $f: L \to M$ be a map which commutes with the operations of G, i.e. $f\{g(x)\} = g\{f(x)\}$ $(x \in L)$. Then the induced homology homomorphism

$$f_*: \bar{H}_n(L, K) \to \bar{H}_n(M, K)$$

is an epimorphism if any one of the following conditions is satisfied:

(a) Let X denote L or M, and let X' be the space obtained from X by identifying x, gx for every $x \in X$, $g \in G$. Let $p: L \to L'$, $q: M \to M'$ be the identification maps. Then $H_n(M'; K)$, $H_n(G, 1; K)$ have the same order.

(b) $\bar{H}_n(M; K)$, $H_{n+1}(G, 1; K)$ have the same order, and $H_{n+1}(M'; K) = 0$.

(c) $\bar{H}_n(M; K) \xrightarrow{q_*} H_n(M'; K)$ maps to zero, where q_* is induced by the identification map q mentioned in (a).

Note that $\bar{H}_n(L; K)$ is the factor group of $H_n(L; K)$ by the subgroup of elements of the form $g_* u - u$, where $u \in H_n(L; K)$ and

$$g_*: H_n(L; K) \to H_n(L; K)$$

is the automorphism induced by $g \in G$. Thus the projection

$$\pi: H_n(L; K) \to \bar{H}_n(L; K)$$

is an epimorphism. Hence, if $f_*: \bar{H}_n(L; K) \to \bar{H}_n(M; K)$ is an epimorphism, then the composite homomorphism

$$f_* \pi: H_n(L; K) \to \bar{H}_n(M; K)$$

is an epimorphism. Similarly, for cohomology

$$\bar{H}^n(L; K) = H^0\{K(G, 1); H^n(L; K)\},$$

which is a subgroup of $H^n(L; K)$. Thus the injection

$$i: \bar{H}^n(L; K) \to H^n(L; K)$$

is a monomorphism. I state the analogue of Theorem 3 for cohomology.

THEOREM 4. *Let L, M be arcwise-connected Hausdorff spaces, G a finite abelian group operating, without fixed points, on L and M. Let*

$$H^i(L, K) = 0, \qquad H^i(M, K) = 0$$

for $i < n$ and K an abelian coefficient group. Let $f: L \to M$ be a map which commutes with the operations of G. Then the induced cohomology homomorphism

$$f^*: \bar{H}^n(M, K) \to \bar{H}^n(L, K)$$

is a monomorphism if any one of the following conditions is satisfied:

(a) Let M', L' be obtained as identification spaces exactly as for homology. Then $H^n(M', K)$, $H^n(G, 1; K)$ have the same order.

(b) $\bar{H}^n(M; K)$, $H^{n+1}(G, 1; K)$ have the same order, and $H^{n+1}(M'; K) = 0$.

(c) $H^n(M'; K) \xrightarrow{q^*} \bar{H}^n(M; K)$ maps to zero, where q^* is induced by the identification map q.

From a previous remark we see that the composite homomorphism $if^*: \bar{H}^n(M; K) \to H^n(L; K)$ is a monomorphism when f^* is a monomorphism.

3. Proof of the main theorem

We first prove some preliminary results on fibre spaces. The term *fibre space* is here used in the sense of Serre (**5**). Consider a fibre space E with fibre F and base B, both arcwise connected. Let $H_i(F; K) = 0$ $(i < n)$ for a given coefficient group K, which I omit for the rest of this paragraph. There is a homology spectral sequence associated with this fibre space in which $E_2^{p,q} \cong H_p\{B; H_q(F)\}$, and the $E_\infty^{p,q}$ give a composition series for $H_{p+q}(E)$. The $H_q(F)$ form a local coefficient system over B. Serre proves the exactness of the following sequence, assuming that the local coefficient system of the homology of the fibre over the base is trivial,

$$H_n(F) \to H_n(E) \to H_n(B) \to H_{n-1}(F) \to \ldots \to H_1(E) \to H_1(B).$$

From his proof it is easy to see that, without any assumptions on the local coefficient system, the following sequence is exact

$$\bar{H}_n(F) \to H_n(E) \to H_n(B) \to \bar{H}_{n-1}(F) \to \ldots \to H_1(E) \to H_1(B).$$

Here

$$\bar{H}_r(F) \cong H_0\{B; H_r(F)\}.$$

Serre considers the $E_2^{p,q}$ in each dimension. When $p+q \leqslant n$, $E_2^{p,q} = 0$ unless p or $q = 0$. Thus the only non-zero terms of the spectral sequence

in dimensions $r \leqslant n$ are $H_r(B)$ and $\bar{H}_r(F)$. The composition series for $H_r(E)$ is simply $0 \subset D^{r,0} \subset H_r(E)$, where

$$D^{r,0} = E_\infty^{r,0}, \qquad H_r(E)/E_\infty^{0,r} \cong E_\infty^{r,0}.$$

Thus the following sequence is exact

$$0 \to E_\infty^{0,r} \to H_r(E) \to E_\infty^{r,0} \to 0. \tag{1}$$

But $$E_\infty^{0,r} = E_2^{0,r}/\operatorname{im} d_{r+1},$$

where $d_{r+1}\colon E_{r+1}^{r+1,0} \to E_{r+1}^{0,r}$ and $r < n$. Thus

$$H_{r+1}(B) \to \bar{H}_r(F) \to H_r(E)$$

is exact. In the case $r = n$ there are three non-zero terms of the spectral sequence in dimension $r+1$, namely $E_r^{n+1,0}$, $E_r^{1,n}$, $E_r^{0,n+1}$. However, again $E_\infty^{0,n} = E_2^{0,n}/\operatorname{im} d_{n+1}$ and $E_{n+1}^{0,n} = E_2^{0,n}$, $E_{n+1}^{n+1,0} = E_2^{n+1,0}$. Thus the above sequence (1) extends to

$$H_{n+1}(B) \to \bar{H}_n(F) \to H_n(E) \to \ldots.$$

The composition series for $H_{n+1}(E)$ is

$$0 \subset D^{0,n+1} \subset D^{n+1,0} \subset H_{n+1}(E),$$

where $$D^{0,n+1} \cong E_\infty^{0,n+1}, \qquad D^{n+1,0}/D^{0,n+1} \cong E_\infty^{1,n},$$

and $$H_{n+1}(E)/D^{n+1,0} \cong E_\infty^{n+1,0},$$

giving the exact sequence

$$0 \to D^{n+1,0} \to H_{n+1}(E) \to E_\infty^{n+1,0} \to 0.$$

From this we see immediately the exactness of the sequence

$$H_{n+1}(E) \to H_{n+1}(B) \to \bar{H}_n(F) \to H_n(E) \to H_n(B) \to \ldots.$$

Thus Serre's sequence has been extended by two terms. A precisely similar sequence and extension exist for cohomology

$$H^{n+1}(E) \leftarrow H^{n+1}(B) \leftarrow \bar{H}^n(F) \leftarrow H^n(E) \leftarrow H^n(B) \leftarrow \ldots.$$

Let us consider the circumstances of Theorem 3. Identify the points of L, M respectively which are images under operations of G, giving identification space L', M' respectively and covering maps $p\colon L \to L'$, $q\colon M \to M'$ as described above. G is the group of covering transformations in each case.

Now given a space X and covering space $\tilde{X}$ with projection p and group π of covering transformations, we can construct a fibre space and hence a spectral sequence as follows [Serre, Adams (**5**), (**1**)]. Embed X in a space X^* such that

$$\pi \cong \pi_1(X^*), \qquad \pi_q(X^*) = 0 \quad (q > 1).$$

Let $\bar{X}$ be the space of paths in X^* starting in X; then X is a deformation

retract of $\overline{X}$, and $\overline{X}$ is a fibre space over X^* with fibre of the same singular homotopy type as $\tilde{X}$; π operates on $H(\tilde{X})$ by covering translations, giving the required local coefficient system. Thus there is a spectral sequence associated with this fibration whose second term is

$$E_2^{p,q} \cong H_p\{K(\pi, 1); H_q(\tilde{X})\},$$

and $E_\infty^{p,q}$ gives a composition series for $H_{p+q}(X)$.

Consider the spectral sequences associated with the coverings

$$p\colon L \to L', \qquad q\colon M \to M'.$$

Since L, M are Hausdorff, these are regular coverings. In the first case

$$E_2^{p,q} \cong H_p\{G, 1; H_q(L)\}$$

and in the second case

$$E_2^{p,q} \cong H_p\{G, 1; H_q(M)\}.$$

The map $f\colon L \to M$ induces a map of these spectral sequences, and, since

$$H_i(L) = 0, \qquad H_i(M) = 0 \quad (i < n),$$

the exact sequence and extension constructed above are valid. The following diagram is obviously commutative. As we have already remarked, the coefficient group is K throughout.

$$\begin{array}{ccccccccc}
H_{n+1}(L') & \longrightarrow & H_{n+1}(G,1;K) & \longrightarrow & \overline{H}_n(L) & \xrightarrow{p_*} & H_n(L') & \xrightarrow{r} & H_n(G,1;K) \to 0, \\
\downarrow & & \cong\downarrow i_{n+1*} & & \downarrow f_* & & \downarrow f'_* & & \cong\downarrow i_{n*} \\
H_{n+1}(M') & \longrightarrow & H_{n+1}(G,1;K) & \longrightarrow & \overline{H}_n(M) & \xrightarrow[q_*]{} & H_n(M') & \xrightarrow[s]{} & H_n(G,1;K) \to 0.
\end{array} \tag{2}$$

(*a*) r, s are both epimorphisms, and s is in fact an isomorphism since $H_n(M')$ and $H_n(G, 1; K)$ are of the same order. But $i_{n*}r = sf'_*$, and $i_{n*}r$ is an epimorphism; therefore f'_* is an epimorphism. Since i_{n+1*} is an epimorphism and i_{n*} is a monomorphism, it follows from the exactness of the sequences that f_* is an epimorphism.

(*b*) Since $H_{n+1}(M') = 0$, the homomorphism $H_{n+1}(G, 1; K) \to \overline{H}_n(M)$ is a monomorphism, and, since both groups are of the same order, it is in fact an isomorphism. By commutativity, the composite

$$H_{n+1}(G, 1; K) \to \overline{H}_n(L) \xrightarrow[f_*]{} \overline{H}_n(M)$$

is an isomorphism. Hence f_* is an epimorphism. It is obvious that this condition implies condition (*a*).

(*c*) This condition is obviously implied by (*b*) and is equivalent to (*a*).

If the local coefficient-system formed by the homology of the fibre is trivial in the spectral sequence associated with M, $\overline{H}_n(M) = H_n(M)$.

We have already observed that $H_n(L) \to \bar{H}_n(L)$ is an epimorphism. Hence $f_*: H_n(L) \to H_n(M)$ is an epimorphism.

The proof of Theorem 4 is precisely similar to the above proof and is easy to see from the cohomology diagram, which is as follows:

$$\begin{array}{ccccccccc} H^{n+1}(L') & \leftarrow & H^{n+1}(G,1;K) & \leftarrow & \bar{H}^n(L) & \leftarrow & H^n(L') & \leftarrow & H^n(G,1;K) \leftarrow 0 \\ \uparrow & & \cong \uparrow i^*_{n+1} & & \uparrow f^* & & \uparrow f'^* & & \cong \uparrow i^*_n \\ H^{n+1}(M') & \leftarrow & H^{n+1}(G,1;K) & \leftarrow & \bar{H}^n(M) & \leftarrow & H^n(M') & \leftarrow & H^n(G,1;K) \leftarrow 0. \end{array} \tag{3}$$

4. Applications

COROLLARY 1. *Let L be an arcwise-connected Hausdorff space such that $H_i(L; Z_m) = 0$ $(i < n)$, where Z_m is the group of integers* (mod m). *Let $\phi: L \to L$, $\psi: S^n \to S^n$ be maps such that $\phi^{(r)}$, $\psi^{(r)}$ (ϕ, ψ iterated r times) have no fixed points if $0 < r < m$ and $\phi^{(m)} = 1$, $\psi^{(m)} = 1$, where $m > 2$. Let ψ be orientation-preserving. Let $f: L \to S^n$ be a map such that $f\phi(x) = \psi f(x)$ for every $x \in L$. Then the induced homomorphism*

$$f_*: H_n(L; Z_m) \to H_n(S^n; Z_m)$$

is an epimorphism.

COROLLARY 2. *If L is an arcwise-connected Hausdorff space such that $H^i(L; Z_m) = 0$ $(i < n)$, and f, S^n, ϕ, ψ are as in Corollary 1, then the induced homomorphism*

$$f^*: H^n(S^n; Z_m) \to H^n(L; Z_m)$$

is a monomorphism.

Here the group G operating on L, S^n is Z_m. Identifying images under ϕ, ψ respectively gives spectral sequences as before. Denote by Q^n the identification space obtained from S^n. Q^n is n-dimensional; hence $H_{n+1}(Q^n; Z_m) = 0$. The diagram in this case is

$$\left.\begin{array}{ccc} H_{n+1}(Z_m, 1; Z_m) & \to & \bar{H}_n(L; Z_m) \\ \downarrow & & \downarrow \\ 0 \to H_{n+1}(Z_m, 1; Z_m) & \to & \bar{H}_n(S^n; Z_m) \end{array}\right\}. \tag{4}$$

Since ψ is orientation-preserving,

$$\bar{H}_n(S^n; Z_m) = H_n(S^n; Z_m).$$

But $$H_n(S^n; Z_m) = H_{n+1}(Z_m, 1; Z_m) = Z_m.$$

Hence condition (b) of Theorem 3 is satisfied. Thus

$$f_*: H_n(L; Z_m) \to H_n(S^n; Z_m)$$

is an epimorphism.

The proof of Corollary 2 is precisely similar to that of Corollary 1.

COROLLARY 3 [cf. Jaworowski (7)]. *Let L be an arcwise-connected Hausdorff space with $H_i(L; Z_2) = 0$ $(i < n)$. Let ϕ be a continuous involution on L. Let $f: L \to S^n$ be a map such that $f\{\phi(x)\} \neq f(x)$ for any $x \in L$. Then the induced homology homomorphism*

$$f_*: H_n(L; Z_2) \to H_n(S^n; Z_2)$$

is an epimorphism.

COROLLARY 4. *If L is an arcwise-connected Hausdorff space with $H^i(L; Z_2) = 0$ $(i < n)$ and ϕ, f are as above in Corollary 3, then the induced cohomology homomorphism*

$$f^*: H^n(S^n; Z_2) \to H^n(L; Z_2)$$

is a monomorphism.

Following Jaworowski we may deform f to a map $g: L \to S^n$ such that the map $\psi: S^n \to S^n$ induced by g from ϕ is in fact the antipodal map on S^n, i.e. $g\{\phi(x)\} = -g(x)$. This is done by putting

$$g(x) = \frac{f(x) - f\{\phi(x)\}}{|f(x) - f\{\phi(x)\}|},$$

for every $x \in L$. Note that $f(x)$, $g(x)$ are not antipodal, so that f and g are homotopic. The proofs of Corollaries 3 and 4 are now exactly similar to those of Corollaries 1 and 2. Note that Z_2 operates trivially on $H_n(S^n)$.

From Corollary 3 we get at once a further generalization of another of Borsuk's theorems mentioned by Jaworowski. I simply state the result.

COROLLARY 5. *Let L, ϕ be as above in Corollary 1. Then for every map f of L into Euclidean n-space E^n, there exists a fixed point $x_0 \in L$; i.e. $f\{\phi(x_0)\} = f(x_0)$.*

5. The relative case

The main theorems may be generalized to a pair of spaces. We use the spectral sequence obtained as follows. Let (X, A) be a pair such that $\pi_1(A) \to \pi_1(X)$ is an epimorphism. Let $\tilde{X}$ be a covering of X with group π of covering translations. Let $\tilde{A}$ be the counter-image of A in $\tilde{X}$. Since $\pi_1(A) \to \pi_1(X)$ is an epimorphism, $\tilde{A}$ is connected. Embed X in X^* such that

$$\pi_1(X^*) \cong \pi, \qquad \pi_q(X^*) = 0 \quad (q > 1).$$

Let $\bar{X}$ be the space of paths in X^* starting in X, $\bar{A}$ the space of paths in X^* starting in A. Then X is a deformation retract of $\bar{X}$, and A of $\bar{A}$. $\bar{X}$ is a fibre space over X^* with fibre of the singular homotopy type of $\tilde{X}$; $\bar{A}$ is a fibre space over X^* with fibre of the singular homotopy type of $\tilde{A}$.

π operates on $H(\tilde{X}, \tilde{A})$ by covering translations, giving the required local coefficient system. Thus there is a spectral sequence whose second term is

$$E_2^{p,q} \cong H_p\{\pi, 1; H_q(\tilde{X}, \tilde{A})\}$$

and such that $E_\infty^{p,q}$ gives a composition series for $H_{p+q}(X, A)$.

Now let L, M be arcwise-connected Hausdorff spaces, and let $P \subset L$, $N \subset M$ be subspaces such that $\pi_1(P) \to \pi_1(L)$, $\pi_1(N) \to \pi_1(M)$ are epimorphisms. Let $f\colon (L, P) \to (M, N)$ be a map, and let

$$H_i(L, P; K) = 0, \qquad H_i(M, N; K) = 0 \quad (i < n).$$

Let the abelian group G operate on (L, P) and on (M, N) without fixed points and such that its operations commute with f.

THEOREM 5. *The induced map*

$$f_*\colon H_n(L, P; K) \to H_n(M, N; K)$$

is an epimorphism if any one of the following conditions is satisfied.

(*a*) $H_n(M', N'; K)$, $H_n(G, 1; K)$ *are both of the same order, where* (M', N') *is obtained from* (M, N) *by identifying images under operations of* G.

(*b*) $\bar{H}_n(M, N; K)$, $H_{n+1}(G, 1; K)$ *have the same order, and*

$$H_{n+1}(M', N', K) = 0.$$

(*c*) $\bar{H}_n(M, N; K) \xrightarrow{q_*} H_n(M', N'; K)$ *maps to zero, where* q_* *is induced by the identification map mentioned in* (*a*).

The proof is obvious from the spectral sequence constructed above.

There is a similar result for cohomology which is the exact analogue of Theorem 4.

In conclusion, I should like to express my gratitude to Professor J. H. C. Whitehead for much valuable criticism and encouragement in the preparation of this note.

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ON THE INTEGRABILITY OF A CLASS OF NON-OSCILLATORY DIFFERENTIAL EQUATIONS BY MEANS OF LAPLACE TRANSFORMS

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1. A FUNCTION $g(t)$, given on an open half-line (t^0, ∞), is called *completely monotone* on (t^0, ∞) if it possesses derivatives of arbitrarily high order satisfying $(-1)^n d^n g(t)/dt^n \geqslant 0$ for $n = 0, 1, \ldots$ and for every t on (t^0, ∞). This will be abbreviated by saying that $g(t)$ is $CM(t^0)$. A theorem of Hausdorff and Bernstein states that $g(t)$ is $CM(t^0)$ if and only if $g(t)$ is representable on (t^0, ∞) in the form

$$\int_0^\infty e^{-ts}\, d\mu(s), \tag{1}$$

where μ is a non-decreasing function for which the integral (1) becomes convergent on (t^0, ∞).

In (**2**) this theorem was applied in order to account for the existence of the Laplacean structure (1) (with *non-negative* '*density*' $d\mu$) [cf. (**4**)] of a class of transcendents, occurring in mathematical physics, which are defined by differential equations of the form

$$x'' + f(t)x = 0. \tag{2}$$

In fact, it was shown in (**2**) that, if $-f(t)$ is $CM(0)$, then (2) possesses a solution $x(t)$ which is representable on $(0, \infty)$ in the form (1), where μ is monotone, and that this solution $x(t)$ of (2) is substantially unique. Here and in the sequel, only real-valued solutions $x(t)$ distinct from the trivial solution $x(t) \equiv 0$ are referred to as solutions of (2). It is also understood that two such solutions $x(t)$ are substantially identical if one of them is a constant multiple of the other (correspondingly, the monotone μ of (1) can be multiplied by any positive or negative constant).

The simplest instance results if $f(t^{-1})$ is any (rational or transcendental) entire function in t having the property that $a_n \leqslant 0$ holds for $n = 0, 1, \ldots$ in the expansion $f(t) = \sum a_n/t^n$; for then every term $-a_n/t^n$ of $-f(t)$, hence $-f(t)$ itself, is $CM(0)$. But not even in this simplest instance of the theorem is the μ belonging to the solution (1) of (2) available explicitly,

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except when the entire function $f(t^{-1})$ is a quadratic polynomial in t. This exceptional case results from the circumstance that (2) is of the confluent hypergeometric type if $a_n = 0$ for every $n > 2$.

2. A theorem of quite a different type will be proved in what follows. On the one hand, not $-f(t)$, but $f(t)$ itself, will be assumed to be $CM(0)$. On the other hand, the Laplacean representation will refer to *all*, rather than (as in the preceding case) to a substantially unique, solution of (2). But, since every function (1), where μ is monotone (and not a constant) is either positive or negative throughout, an additional condition will be needed since it will be necessary to assure that the solutions $x(t)$ of (2) do not have zeros which cluster at $t = \infty$.

In view of Sturm's separation theorem, this condition is satisfied by every solution $x(t)$ of (2) if it is satisfied by a single solution $x(t)$. Hence the property in question, the so-called *non-oscillatory* character of (2), is a property of $f(t)$ alone. According to Sturm's comparison theorem, the case $f = f_1$ of (2) is non-oscillatory whenever the case $f = f_2$ of (2) is non-oscillatory for some f_2 satisfying $f_1 \leqslant f_2$ for all large t. Since $x'' = 0$, the case $f = 0$ of (2), is non-oscillatory, it follows that (2) is non-oscillatory if $f(t) \leqslant 0$, and so in particular if $-f(t)$ is $CM(0)$. This is the reason why in the theorem quoted in § 1 it was not necessary to make the non-oscillatory character of (2) an additional assumption. That the non-oscillatory character of (2) can be realized when $f(t)$ itself is $CM(0)$ (hence, in particular, if $f(t) \geqslant 0$), is shown by the example $f(t) = c/t^2$, where $0 \leqslant c \leqslant \frac{1}{4}$. If this example, which will be discussed in a moment, is combined with Sturm's comparison theorem, it follows that all conditions to be required of $f(t)$ are satisfied if $f(t)$ is $CM(0)$ and $4t^2 f(t) \leqslant 1$ holds on $(0, \infty)$.

The general theorem, to be proved in §§ 3–4 and to be applied to particular cases in § 5, is as follows:

Let $f(t)$ be $CM(0)$ (hence $f(t) \geqslant 0$), and let (2) be non-oscillatory. If $x(t)$ is any (real-valued, non-trivial) solution of (2) on $(0, \infty)$, let $t_0 = 0$ or $0 < t_0 < \infty$ according as $x(t)$ does not or does have at least one zero on $(0, \infty)$, and let t_0 be the last zero of $x(t)$ in the second case. Then the function

$$y(t) = 1/x(t) \tag{3}$$

is representable on (t_0, ∞) in the form (1), where $\mu(s)$ is monotone (and such as to render the integral (1) convergent for every $t > t_0$).

It is understood that $\mu(s)$ is non-decreasing or non-increasing throughout according as $x(t) > 0$ or $x(t) < 0$ on (t_0, ∞), and that $\mu(s) - \mu(0)$, when

normalized by $\mu(s+0) = \mu(s)$ for $0 \leqslant s < \infty$, is uniquely determined by the (arbitrary) solution $x(t)$ of (2).

As an illustration, choose $f(t) = c/t^2$, where c is a constant. Then $f(t)$ is $CM(0)$ if $0 \leqslant c$, and (2) is non-oscillatory if $0 \leqslant c \leqslant \frac{1}{4}$ (A. Kneser). In fact, if $x(t) = t^b$, where $b = b(c)$, is tried for the case $f = c/t^2$ of (2), there results for b the equation $b(b-1)+c = 0$. Since the latter has real roots (and $x(t) = t^b$ is, therefore, positive for all $t > 0$) if $0 \leqslant c \leqslant \frac{1}{4}$, the assertion follows. Since the roots b cover the interval $0 \leqslant b \leqslant 1$ when $0 \leqslant c \leqslant \frac{1}{4}$, the theorem to be proved is illustrated by the fact that the function $x(t) = 1/t^\epsilon$ is $CM(0)$ for arbitrarily small values of $\epsilon > 0$. Since not only $t^{\frac{1}{2}}$ but also $t^{\frac{1}{2}} \log t$ is a solution if $c = \frac{1}{4}$, it also follows that, if a is constant, then $t^{-\frac{1}{2}}/(a+\log t)$ or $t^{-\frac{1}{2}}(1+a \log t)$ is $CM(t_0)$ whenever it is positive on (t_0, ∞). Corresponding results follow by taking linear combinations of two solutions of (2) in the general case.

3. The proof of the italicized theorem proceeds as follows.

Since $-x(t)$ is a solution of (2) if $x(t)$ is, and since (2) is non-oscillatory, it can be assumed that $x(t) > 0$ holds on (t_0, ∞), where $t_0 \geqslant 0$. In view of (3), this means that if $n = 0$, then it is true that

$$(-1)^n d^n y(t)/dt^n \geqslant 0 \quad \text{on } (t_0, \infty). \tag{4_n}$$

What is at stake is the truth of (4_n) for every n. In fact, the proof of the theorem can then be concluded by an appeal to the Hausdorff–Bernstein theorem.

First, since $f(t)$ is $CM(0)$ and hence $CM(t_0)$, not only $x(t) > 0$ but also $f(t) \geqslant 0$ holds on (t_0, ∞). It follows therefore from (2) that $x''(t) \leqslant 0$ on (t_0, ∞). This means that, over the entire half-line (t_0, ∞), the graph of $x = x(t) > 0$ turns its concavity toward the t-axis (with the understanding that the graph might contain straight segments). Clearly, this is possible only if $x'(t) \geqslant 0$ on (t_0, ∞). This, when combined with $x(t) > 0$, means that, if $r(t)$ denotes the logarithmic derivative

$$r(t) = x'(t)/x(t) \tag{5}$$

(which exists on (t_0, ∞), since $x(t)$ does not vanish there), then $r(t) \geqslant 0$ on (t_0, ∞).

Next, division of (2) by $x(t)$ shows that (5) is a solution of Riccati's differential equation

$$r' + r^2 + f(t) = 0 \tag{6}$$

on (t_0, ∞). But, since $f(t) \geqslant 0$, it is clear from (6) that $r'(t) \leqslant 0$ on (t_0, ∞).

Finally, since $(-1)^n d^n f(t)/dt^n \geqslant 0$ is assumed for $n = 0, 1, \ldots$ on $(0, \infty)$, and therefore on (t_0, ∞), and since $r(t) \geqslant 0$ and $r'(t) \leqslant 0$ on (t_0, ∞), successive differentiations of (6), when combined with an induction, show

that $(-1)^n d^n r(t)/dt^n \geqq 0$ holds on (t_0, ∞) not only for $n = 0$ and $n = 1$ but for $n = 2, 3, \ldots$ also [cf. (3), 591]. Accordingly, $r(t)$ is $CM(t_0)$.

4. It will now be shown that (4_n), as it stands, is true for every n *because* (4_n) becomes true for every n if $y(t)$ is replaced by $r(t)$ (and because (4_0), as it stands, is true). This will complete the proof of the theorem. The proof will depend on an induction having the type of a Euclidean algorithm.

The algorithm in question is the recursion formula

$$x(t)g_{n+1}(t) = r(t)g_n(t) - g_n'(t) \tag{7}$$

on (t_0, ∞), with

$$g_0(t) = 1. \tag{8}$$

Since sums and products of functions which are $CM(t_0)$ are $CM(t_0)$, and since $-g'(t)$ is $CM(t_0)$ if $g(t)$ is, it is clear from (7) that $g_{n+1}(t)$ is $CM(t_0)$ if $g_n(t)$ is. It follows therefore from (8) that every $g_n(t)$ is $CM(t_0)$.

In particular, both $g_n(t)$ and $-g_n'(t)$ are non-negative on (t_0, ∞). Since the same is true of $r(t)$ and $x(t)$, it follows from (7) that $g_{n+1}(t) \geqq 0$ holds on (t_0, ∞) for $n+1 \geqq 1$ and, in view of (8), for $n+1 = 0$ also. Hence, in order to complete the proof of (4_n) for every n, it is sufficient to verify that, on (t_0, ∞),

$$(-1)^n d^n y(t)/dt^n = g_n(t)/x(t), \tag{9$_n$}$$

since $x(t) > 0$.

To this end, define $g_0(t)$, $g_1(t)$,... on (t_0, ∞) by (9_n), rather than by (7) and (8). Since (3) shows that (9_n) reduces to (8) if $n = 0$, the identity of the two definitions of $g_n(t)$ will follow if it is verified that (7) is implied by (9_n) and (9_{n+1}). But, if (9_n) is differentiated and the result is compared with (9_{n+1}), it is seen that $g_{n+1}(t)$ is identical with the derivative of $-g_{n+1}(t)/x(t)$. Accordingly

$$g_{n+1}' = -(g_n' x - g_n x')/x^2 \tag{10}$$

on (t_0, ∞), where $x > 0$. Since (10) and (5) imply (7), the proof of the italicized theorem is now complete.

5. Two applications, (α) and (β), of the theorem will now be given.

(α) The theorem is applicable whenever the coefficient function of (2) is of the form

$$f(t) = \int_0^\infty e^{-ts} p(s)\, ds \tag{11}$$

if $p(s)$ is any function satisfying the following three conditions: $p(s)$ is (a) integrable on every finite s-interval, (b) non-negative throughout, and (c) not greater than $\frac{1}{4}s$.

In fact, since (*c*) implies that $p(s) = O(s)$ as $s \to \infty$, it is clear from (*a*) and (*b*) that (11) represents a completely monotone function $f(t)$ on the half-line $(0, \infty)$. Hence (α) follows if it is ascertained that (2) is non-oscillatory. In view of the second paragraph of § 2, this condition is certainly satisfied if $4t^2 f(t) \leqslant 1$ (actually, this is not needed on the whole of $(0, \infty)$, but only for large t). Finally, this inequality, and even $4f(t) < t^{-2}$, follows from (*c*) since the integral (11) becomes the function t^{-2} if $p(s) = s$ and $t > 0$. This proves (α).

(β) Let a function $F(z) = b_0 + b_1 z + \ldots$ be regular in a sufficiently small circle about $z = 0$, suppose that $b_m \geqslant 0$ for every m and that $b_0 = 0$, $b_1 = 0$, and $b_2 < \frac{1}{4}$, and define $f(t)$, for large positive t, by placing $f(t) = F(z^{-1})$. Then (2) has the following property: *corresponding to every solution $x(t)$ (which is real-valued and distinct from the trivial solution $x(t) \equiv 0$), there exist a sufficiently large t_0 and a monotone function $\mu(s)$ having the property that $1/x(t)$ is representable on the half-line (t_0, ∞) in the form* (1).

In order to prove (β), note that, since

$$f(t) = \sum_{m=2}^{\infty} b_m / t^m \tag{12}$$

for large t, it follows from $b_2 < \frac{1}{4}$ that $\limsup 4t^2 f(t) < 1$, where $t \to \infty$. Hence (2) is non-oscillatory on (T, ∞) if $T > 0$ is large enough. On the other hand, since $b_m \geqslant 0$, the function (12) is $CM(T)$ (here T is chosen so large that $F(z)$ is regular for $|z| < 1/T$). This proves (β) since it is clear that the general italicized theorem remains true if $t = 0$ is replaced by any $t = T$.

Since the particular case (β) of the general theorem seems to be a substantially *formal* statement, it could perhaps be verified directly, that is, by a comparison of like powers of $1/t$ [in this connexion, cf., e.g. (**1**), 309–24]. But I did not succeed with such a straightforward proof of (β).

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ON THE EXTENSION OF A THEOREM OF ATKINSON'S

By JOHN JONES, Jr. (*University of Tennessee*)

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F. V. ATKINSON (**1**) considered the non-linear differential equation

$$y''+f(x)y^{2n-1} = 0 \tag{1}$$

and obtained a necessary and sufficient condition that all solutions be oscillatory, where n is a positive integer not less than 2, $f(x)$ a positive real-valued continuous function of a real variable x for non-negative x. An oscillatory solution is a non-identically zero solution which possesses an infinite number of zeros for $x \geqslant 0$. The purpose of this paper is to obtain a necessary and sufficient condition for the oscillation of all solutions of the equation

$$y'' + \sum_{i=2}^{n} f_i(x)y^i = 0 \tag{2}$$

by use of a similar method. Equations of this type arise in astrophysics and special cases have been studied by R. H. Fowler (**2**). Only real non-identically zero solutions will be considered. C. T. Taam (**3**), (**4**) has shown that, if $f_i'(x)$ are L-integrable in $[a,\infty)$ $(i = 1, 2,..., n)$, $f_i(x) \geqslant m > 0$ in $[a,\infty)$, then every solution $y(x)$ of (2) is bounded in $[a,\infty)$.

We have the following results.

THEOREM. *Let $f_i(x)$ be non-negative and continuous for $x \geqslant 0$ on $[a,\infty)$ $(i = 2, 3,..., n)$, $f_k(x) > 0$ for some index k, and $f_i'(x)$ be L-integrable in $[a,\infty)$, then all solutions of* (1.2) *are oscillatory if and only if*

$$\int_0^\infty \sum_{i=2}^{n} x f_i(x)\, dx = \infty. \tag{3}$$

No solution $y(x)$ of (2) becomes infinite for any positive finite value of x. Let $y(x)$ be a non-oscillatory solution of (2); then we have to show that (3) cannot hold. Since, by assumption, $y(x)$ can have only a finite number of zeros for $x \geqslant 0$, $y(x)$ will be either positive or negative for $x \geqslant a$, where a is some non-negative real number. Let us assume $y(x)$ to be positive for $x \geqslant a$. Now from (2) it is seen that $y''(x) < 0$ for $x \geqslant a$. Thus $y'(x)$ will tend to a positive limit, to zero, to a negative

limit, or to $-\infty$ for increasing values of $x \geqslant a$. But $y'(x)$ cannot tend to a negative limit or to $-\infty$ since that would imply that y is ultimately negative; so $y(x)$ must be a monotone increasing function of x for $x \geqslant a$, and $y'(x)$ tends to a finite non-negative limit.

Upon integrating (2) over the interval $(0, x)$ we obtain the equation

$$y'(x)-y'(0)+\int_0^x \sum_{i=2}^n f_i(t)y^i(t)\,dt = 0. \tag{4}$$

Since $y'(x)$ tends to a finite limit as $x \to \infty$, the integral in (4) converges as $x \to \infty$. Hence integrating (2) over the interval (x, ∞) we get

$$y'(\infty)-y'(x)+\int_x^\infty \sum_{i=2}^n f_i(t)y^i(t)\,dt = 0. \tag{5}$$

Now $y'(\infty) \geqslant 0$, and so

$$y'(x) \geqslant \int_x^\infty \sum_{i=2}^n f_i(t)y^i(t)\,dt. \tag{6}$$

But $y(x) \geqslant 0$ for $x \geqslant a$; thus integrating (6) over the interval (a, x) we have, for $x > a$,

$$\begin{aligned} y(x)-y(a) &\geqslant \int_a^x \int_u^\infty \sum_{i=2}^n f_i(t)y^i(t)\,dt\,du \\ &= \int_a^x (t-a)\sum_{i=2}^n f_i(t)y^i(t)\,dt + (x-a)\int_x^\infty \sum_{i=2}^n f_i(t)y^i(t)\,dt. \end{aligned} \tag{7}$$

From (7) we have

$$y(x) \geqslant \int_a^x (t-a)\sum_{i=2}^n f_i(t)y^i(t)\,dt, \tag{8}$$

i.e.

$$y(x)\left\{\int_a^x (t-a)\sum_{i=2}^n f_i(t)y^i(t)\,dt\right\}^{-1} \geqslant 1. \tag{8}'$$

Since

$$\sum_{i=2}^n f_i(t)y^i(t) \geqslant f_j(t)y^j(t) \geqslant 0,$$

then raising both sides of (8)′ to the jth power and summing on j from 2 to n we have

$$\sum_{j=2}^n \left[(t-a)\sum_{i=2}^n f_i(t)y^i(t)\right]\left[\int_a^x (t-a)\sum_{i=2}^n f_i(t)y^i(t)\,dt\right]^{-j} \geqslant \sum_{j=2}^n (t-a)f_j(t). \tag{9}$$

Integrating (9) over any interval (x_1, x_2) such that $a < x_1 < x_2$ we have

$$\sum_{j=2}^{n}(1-j)^{-1}\Big[\int_a^x (t-a)\sum_{i=2}^{n} f_i(t)y^i(t)\,dt\Big]^{1-j}\Big|_{x_1}^{x_2} \geqslant \int_{x_1}^{x_2}\sum_{j=2}^{n}(t-a)f_j(t)\,dt. \quad (10)$$

Let $x_2 \to \infty$; then the left-hand side of (10) tends to a finite limit. This implies that the integral on the right of (10) is bounded, and so

$$\int_0^\infty \sum_{i=2}^{n} x f_i(x)\,dx < \infty. \quad (11)$$

Hence (3) is sufficient.

Next to establish the necessity. Let us assume that (11) holds and then show that there exists a non-oscillatory solution $y(x)$ of (2). For any prescribed value of $y(\infty)$, for example 1, there exists a solution of (2) such that $y(\infty) = 1$ and $y'(\infty) = 0$. This solution is not oscillatory. If the integral equation

$$y(x) = 1 - \int_x^\infty (t-x)\sum_{i=2}^{n} f_i(t)\{y(t)\}^i\,dt \quad (12)$$

has a solution $y(x)$ which is continuous and uniformly bounded as $x \to \infty$, then it is also a solution of (2) with conditions $y(\infty) = 1$ and $y'(\infty) = 0$. I shall use Picard's method of successive approximations to establish the existence of a bounded continuous solution of (2). Let $\{y_m(x)\}$ $(m = 0, 1, 2,\ldots)$ for $x \geqslant 0$ be a sequence of functions such that

$$y_0(x) \equiv 0, \qquad y_{m+1}(x) = 1 - \int_x^\infty (t-x)\sum_{i=2}^{n} f_i(t)\{y_m(t)\}^i\,dt. \quad (13)$$

If x is so large that

$$\int_x^\infty (t-x)\sum_{i=2}^{n} f_i(t)\,dt < 1 \quad (14)$$

and if (11) holds, then by induction it can be shown that

$$0 \leqslant y_m(x) \leqslant 1 \quad (m = 0, 1, 2,\ldots). \quad (15)$$

Now

$$y_{m+2}(x) - y_{m+1}(x) = \int_x^\infty (t-x)\sum_{i=2}^{n} f_i(t)[\{y_m(t)\}^i - \{y_{m+1}(t)\}^i]\,dt, \quad (16)$$

and, for sufficiently large x, we have

$$|y_{m+2}(x) - y_{m+1}(x)| \leqslant n\Big\{\max_{t \geqslant x}|y_m(t) - y_{m+1}(t)|\Big\}\int_0^\infty (t-x)\sum_{i=2}^{n} f_i(t)\,dt, \quad (17)$$

and the convergence of the sequence $\{y_m(x)\}$ $(m = 0, 1, 2,...)$ for x sufficiently large so that

$$n \int_x^\infty (t-x) \sum_{i=2}^{n} f_i(t)\, dt < 1. \tag{18}$$

The continuity of the limiting function can also be established. Thus the assumption that (11) holds implies the existence of a non-oscillatory solution of (2) and hence (3) is necessary.

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ON THE UPPER AND LOWER LIMITING POINTS OF THE SPECTRUM OF A PARTIAL DIFFERENTIAL EQUATION

By A. I. MARTIN (*Derby*)

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1. THE spectrum of a differential equation

$$\frac{\partial^2\phi(x,y)}{\partial x^2}+\frac{\partial^2\phi(x,y)}{\partial y^2}+\{\lambda-q(x,y)\}\phi(x,y)=0, \tag{1.1}$$

where $q(x,y)$ is a continuous function, λ a variable parameter, and x, y vary over the whole plane, may be regarded as being defined by the various terms which occur in the corresponding expansion, or Parseval formula. More precisely, if $G(x,y,\xi,\eta,\lambda)$ is Green's function for the problem, and if

$$H(x,y,\xi,\eta,\mu)=\lim_{\delta\to 0}\int_0^{\mu}\operatorname{im} G(x,y,\xi,\eta,u+i\delta)\,du,$$

then a point μ lying on the real axis does not belong to the spectrum if it lies inside an open interval throughout which $H=H(\mu)$ is constant. All other points on the real axis are said to form the *spectrum* of (1.1) and the whole plane. The spectrum need not be unique if Green's function is not unique.

For any $f=f(x,y)$ which is L^2 over the whole plane, it has been shown in (**1**) that

$$J(f,\mu)=\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H(x,y,\xi,\eta,\mu)f(x,y)f(\xi,\eta)\,dxdyd\xi d\eta$$

exists and is a non-decreasing function of μ, and that the Parseval formula is given by

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f^2\,dxdy=J(f,\infty)-J(f,-\infty). \tag{1.2}$$

A related formula has also been obtained by considering the integral

$$D(f)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+qf^2\right\}dxdy. \tag{1.3}$$

Quart. J. Math. Oxford (2), 7 (1956), 310-15.

The result is that, under more restrictive conditions,

$$D(f) = \int_{-\infty}^{\infty} \mu \, dJ(f, \mu). \tag{1.4}$$

For example, conditions sufficient for (1.4) to hold are that f should satisfy the same conditions as are required in the expansion formula, and that each term in the integrand on the right-hand side of (1.3) should be separately integrable over the whole plane.

In the case of an ordinary differential equation, many results are already known about the nature of the spectrum. For instance, if Green's function is unique (Weyl's limit-point case), the upper limiting point of the spectrum is $+\infty$, and the lower limiting point lies between the upper and lower limits of the function q at infinity. Here I shall derive corresponding results for (1.1) and the whole plane.

2. Suppose that the spectrum is bounded above. Then by a simple translation of the real λ-axis we may suppose that the spectrum lies wholly to the left of the origin. Thus, $J(f, \mu)$ is constant for $\mu > 0$, and (1.4) gives $D(f) \leqslant 0$.

If q takes positive values somewhere, then clearly we may choose a function f satisfying the conditions for which (1.4) holds and such that $D(f) > 0$. This is a contradiction.

If $q \leqslant 0$ everywhere, then we consider the function given by

$$f = (R^2 - x^2 - y^2)^3 \quad \text{inside the circle } x^2 + y^2 = R^2,$$

$$f = 0 \quad \text{outside this circle.}$$

This function satisfies the conditions for which (1.4) holds. Hence

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 \right\} dx dy \leqslant \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (-q) f^2 \, dx dy$$

$$\leqslant Q(R) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f^2 \, dx dy,$$

where $Q(R)$ is the maximum value of $-q$ inside the above circle. Evaluation of the integrals then yields $Q(R) \geqslant 42/5R^2$, which leads to a contradiction on making $R \to 0$.

We thus have the following result:

Whether Green's function is unique or not, the spectrum corresponding to (1.1) *and the whole plane is not bounded above.*

3. In this section I shall suppose *Green's function to be unique*. Let l be the greatest number such that only a finite number of points of the spectrum lie below $l-\epsilon$ for every positive ϵ, each such point having finite degeneracy. If the spectrum is not bounded below, then we take $l = -\infty$. If $q(x, y)$ is not less than a constant k for all sufficiently large values of x^2+y^2, then it has been shown by Titchmarsh (**2**) that there are only a finite number of points of the spectrum in the interval $\mu < k$, each point having finite degeneracy. Thus,

$$l \geqslant \liminf_{x^2+y^2 \to \infty} q(x, y). \tag{3.1}$$

Conversely, suppose that there are only a finite number of points of the spectrum of finite degeneracy in the interval $\mu < k$. Let these points be given by $-\infty < \lambda_1 < \lambda_2 < \ldots < \lambda_N < k$. For simplicity, we suppose that $k = 0$. This situation can always be obtained by making a simple translation. In $\lambda_r < \mu < \lambda_{r+1}$, $J(f, \mu)$ is constant; whilst at λ_r it has the saltus c_r^2 (or a *finite* sum of such terms in the case of degeneracy), where

$$c_r = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f \psi_r \, dxdy$$

is the 'Fourier coefficient' of f with respect to the eigenfunction

$$\psi_r = \psi_r(x, y)$$

corresponding to the eigenvalue λ_r. The formula (1.4) now gives

$$D(f) \geqslant \sum_{r=1}^{N} \lambda_r c_r^2 > -A \sum_{r=1}^{N} c_r^2, \tag{3.2}$$

where A is a constant.

Consider the function given by

$$f = (b-r)^3(r-a)^3 \quad \text{for } 0 < a \leqslant r \leqslant b,$$

$$f = 0 \quad \text{for } r < a \text{ and } r > b,$$

where $r^2 = x^2+y^2$. This function satisfies the conditions for which (1.4) holds. We have

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 \right\} dxdy = 2\pi \int_a^b \left(\frac{df}{dr}\right)^2 r \, dr$$

$$= 18\pi \int_a^b (b-r)^4(r-a)^4(b+a-2r)^2 r \, dr$$

$$= (b-a)^{11}\{c_1 a + c_2(b-a)\},$$

where c_1 and c_2 are constants independent of a and b. Also, if

$$M(a,b) = \max_{a \leqslant r \leqslant b} q,$$

then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} qf^2\,dxdy \leqslant M(a,b)2\pi \int_a^b f^2 r\,dr$$

$$= 2\pi M(a,b)\int_a^b (b-r)^6(r-a)^6 r\,dr$$

$$= M(a,b)(b-a)^{13}\{c_3 a + c_4(b-a)\},$$

where c_3 and c_4 are non-zero constants. Finally, by Schwarz's inequality,

$$c_r^2 \leqslant \left(\iint_{a\leqslant r\leqslant b} f^2\,dxdy\right)\left(\iint_{a\leqslant r\leqslant b} \psi_r^2\,dxdy\right)$$

$$= (b-a)^{13}\{c_3 a + c_4(b-a)\}\iint_{a\leqslant r\leqslant b} \psi_r^2\,dxdy.$$

It thus follows from (3.2) that

$$(b-a)^{11}\{c_1 a + c_2(b-a)\} + M(a,b)(b-a)^{13}\{c_3 a + c_4(b-a)\}$$

$$> -A(b-a)^{13}\{c_3 a + c_4(b-a)\}\sum_{r=1}^{N}\iint_{a\leqslant r\leqslant b}\psi_r^2\,dxdy,$$

i.e. $$M(a,b) > -\frac{c_1 a + c_2(b-a)}{(b-a)^2\{c_3 a + c_4(b-a)\}} - A\sum_{r=1}^{N}\iint_{a\leqslant r\leqslant b}\psi_r^2\,dxdy.$$

As a and b both tend to infinity, the second term on the right-hand side tends to zero since ψ_r is L^2. The first term will tend to zero provided that b tends to infinity much faster than a. It thus follows that

$$\limsup q \geqslant 0,$$

and, generally, $\limsup q \geqslant k$. Making $k \to l$, we then obtain

$$l \leqslant \limsup_{x^2+y^2\to\infty} q(x,y). \tag{3.3}$$

Combining (3.1) with (3.3) we have the following result:

If q tends to a limit as $x^2+y^2 \to \infty$, then this limit is also the lower limiting point of the spectrum.

4. In this section I shall suppose *Green's function to be not unique.* It will be shown that the spectrum cannot be bounded below both for finite and infinite regions.

Integrating by parts and writing $x = r\cos\theta$, $y = r\sin\theta$, we have

$$D_R(f) \equiv \iint_{r\leqslant R} \left\{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + qf^2\right\} dxdy$$

$$= \int_0^{2\pi} f\frac{\partial f}{\partial R} R\, d\theta + \iint_{r\leqslant R} ff^*\, dxdy,$$

where $f^* = (q - \partial^2/\partial x^2 - \partial^2/\partial y^2)f$. If, for all large values of R,

$$\int_0^{2\pi} f\frac{\partial f}{\partial R} R\, d\theta > \text{a constant } k,$$

then integration over $R_1 \leqslant R \leqslant R_2$ gives

$$\int_0^{2\pi} \{f(R_2)^2 - f(R_1)^2\}\, d\theta > 2k\log\frac{R_2}{R_1},$$

and a further integration gives

$$\int_{S_1}^{S_2}\int_0^{2\pi} R_2 f(R_2)^2\, d\theta dR_2 > 2k\int_{S_1}^{S_2} R_2\log\frac{R_2}{R_1} dR_2$$

$$\to \infty,$$

as $S_2 \to \infty$. It follows that, if f and f^* are both L^2 over the whole plane, then

$$D_R(f) \to \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ff^*\, dxdy,$$

as $R \to \infty$ through some sequence of values. Thus, by § 5 of (**1**),

$$\int_{\lambda_0}^{\infty} \mu\, dJ(f,\mu) \leqslant \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ff^*\, dxdy, \tag{4.1}$$

whenever the spectrum is bounded below at λ_0.

It is known that in the case when Green's function is not unique, (1.1) has a non-null solution which is L^2 over the whole plane for any complex value of λ. It is also possible to continue this solution analytically so as to obtain a non-null solution which is L^2 over the whole plane for any real value of λ not lying in the spectrum.† Take the function f in (4.1) to be such a solution for $\lambda < \lambda_0$. Then

$$\int_{\lambda_0}^{\infty} \mu\, dJ(f,\mu) \leqslant \lambda\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^2\, dxdy = \lambda\int_{\lambda_0}^{\infty} dJ(f,\mu),$$

† This will be considered elsewhere. See Theorem 2 of 'Some further work on L^2-solutions of the wave equation' (pp. 280–6 above).

by the Parseval formula (1.2). Since $J(f,\mu)$ is a non-decreasing function, we thus obtain

$$0 \leqslant (\lambda_0-\lambda)\int_{\lambda_0}^{\infty} dJ(f,\mu) \leqslant \int_{\lambda_0}^{\infty} (\mu-\lambda)\, dJ(f,\mu) \leqslant 0.$$

This is possible only if $J(f,\mu)$ is a constant, and (1.2) then shows that f is a null solution. It follows that the spectrum cannot be bounded below.

When Green's function is not unique, the only limit to which q can tend is $-\infty$. Thus the result stated at the end of the last section also holds in the case considered here.

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SOME COMBINATORIAL RESULTS FOR MATRIX POWERS

By P. WHITTLE (*Applied Mathematics Laboratory, D.S.I.R., New Zealand*)

[Received 22 September 1956]

1. I SHALL use the method of generating functions to deduce some results concerning matrix powers which have particular application in the theory of Markov (probability) chains.

The basic result is contained in Theorem 1, which states that, if a matrix A is dominantly diagonal, then $|A|^{-1}$ is equal to the absolute term in the multiple Laurent expansion of a certain simple function. From this is in turn derived a combinatorial formula, displayed in Theorem 2, for the composition of the elements of a matrix power B^N in terms of the elements of B. Theorem 3 provides asymptotic evaluations of certain product sums associated with the elements of B^N.

2. Consider the function

$$F(z_1, z_2, ..., z_p) = \prod_{j=1}^{p} \Big(\sum_{k=1}^{p} a_{jk} z_k / z_j \Big)^{-1} \tag{1}$$

of the complex variables $z_1, z_2, ..., z_p$. In the complex z-space there will be regions D specified by

$$\alpha_j < |z_j| < \beta_j \quad (\alpha, \beta \text{ real};\ j = 1, 2, ..., p) \tag{2}$$

such that F is analytic in all its arguments in D, and may be expanded there in an absolutely convergent multiple Laurent series whose coefficients can be represented by contour integrals of a form analogous to those valid for the classical one-variable case [(**2**), 90–91)].

THEOREM 1. *If the elements a_{jk} of the $p \times p$ matrix A are real and satisfy the relations*

$$a_{jj} > \sum_{k \neq j} |a_{jk}| \quad (j = 1, 2, ..., p), \tag{3}$$

then the absolute term in the Laurent expansion of $F(z_1, z_2, ..., z_p)$ on $|z_1| = |z_2| = ... = |z_p|$ equals $|A|^{-1}$.

Proof. By an expansion on $|z_1| = |z_2| = ... = |z_p|$ I mean an expansion in

$$K(1-\delta) < |z_j| < K(1+\delta), \tag{4}$$

where K is arbitrary (since F is homogeneous of order zero in the z's) and δ is small enough to ensure that the region contains no singularities of F.

For convenience I shall denote a determinant by its diagonal elements: thus

$$\begin{vmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & \cdot & a_{nn} \end{vmatrix} = (a_{11}a_{22}\ldots a_{nn}). \tag{5}$$

We have now, if j and k are distinct,

$$\begin{aligned} (a_{jj}a_{kk}) &\geqslant a_{jj}a_{kk} - |a_{jk}| \cdot |a_{kj}| \\ &> \sum_{l \neq j,k} a_{kk}|a_{jl}| + a_{kk}|a_{jk}| - |a_{jk}| \cdot |a_{kj}| \\ &> \sum_{l \neq j,k} (a_{kk}|a_{jl}| + |a_{jk}| \cdot |a_{kl}|) \\ &\geqslant \sum_{l \neq j,k} |(a_{jl}a_{kk})|. \end{aligned} \tag{6}$$

The absolute term in the Laurent expansion of F is

$$C_0 = \frac{1}{(2\pi i)^p} \oint \cdots \oint \prod_{j=1}^{p} \left(\frac{dz_j}{\sum_k a_{jk}z_k} \right), \tag{7}$$

where the contour for the integration in z_j may be taken as being $|z_j| = 1$.

Now, in virtue of condition (3), the only z_1-singularity of the integrand inside $|z_1| = 1$ is a simple pole at

$$z_1 = -\sum_{k \neq 1} \frac{a_{1k}z_k}{a_{11}}. \tag{8}$$

Performing the z_1-integration in (7) we thus obtain

$$C_0 = \frac{a_{11}^{p-2}}{(2\pi i)^{p-1}} \oint \cdots \oint \prod_{j=2}^{p} \left(\frac{dz_j}{\sum_{k=2}^{p} (a_{11}a_{jk})z_k} \right). \tag{9}$$

By use of (6), the same argument can be repeated for z_2, and so on until we finally obtain a product of factors for C_0^{-1} which is identical with the product obtained in evaluating $|A|$ by pivotal condensation [(1), 45]: that is

$$C_0 = |A|^{-1}, \tag{10}$$

as claimed.

A derivation which seems more elegant at first sight is to make the transformation

$$y_j = \sum_k a_{jk}x_k, \tag{11}$$

so that (7) becomes

$$C_0 = \frac{|A|^{-1}}{(2\pi i)^p} \oint \cdots \oint \prod_j \left(\frac{dy_j}{y_j} \right). \tag{12}$$

To justify this argument, however, one must be able to prove that

the transformation from the x's to the y's, and the simultaneous deformation of the integration contours to suitable paths such as $|y_j| = 1$, can be carried out in a *continuous* fashion during which the transformation is at no time singular, and no singularity of the integrand crosses the integration surface. This justification is direct, but lengthy.

3. Consider an arbitrary square matrix B. The jkth element of a positive integral power B^N of this matrix can be written

$$b_{jk}^{(N)} = \sum_{n_{11}} \sum_{n_{12}} \dots \sum_{n_{pp}} C_{jk}(n_{rs}) \prod_{rs} b_{rs}^{n_{rs}}, \tag{13}$$

where $C_{jk}(n_{rs})$ is used to denote a numerical coefficient which is a function of j, k, n_{11}, n_{12},..., n_{pp}. I shall use the result of Theorem 1 to find an explicit expression for the combinatorial coefficient $C_{jk}(n_{rs})$. This coefficient can be interpreted as the number of distinct ways in which n_{11} quantities b_{11}, n_{12} quantities b_{12},... can be set in order so that the first subscript of any b is the same as the second subscript of the b preceding it, while the first and last subscripts of the whole sequence are j and k respectively. The formula derived for C also has an application in the theory of Markov chains which is described in (3).

I shall use the notation

$$n_{r\cdot} = \sum_{s} n_{rs},$$

$$n_{\cdot s} = \sum_{r} n_{rs}. \tag{14}$$

THEOREM 2. *If for a particular set of n_{rs} we define a matrix*

$$[T_{jk}(n_{rs})] = \operatorname{adj}\left[\delta_{jk} - \frac{n_{jk}}{n_{j\cdot}}\right], \tag{15}$$

then

$$C_{jk}(n_{rs}) = T_{jk}(n_{rs}) \frac{\prod_{r} n_{r\cdot}!}{\prod_{rs} n_{rs}!}, \tag{16}$$

provided that

$$\sum_{r} \sum_{s} n_{rs} = N, \tag{17}$$

$$n_{r\cdot} - n_{\cdot r} = \delta_{rj} - \delta_{rk}, \tag{18}$$

$$n_{r\cdot} > 0, \tag{19}$$

where j, k, $r = 1, 2, \dots, p$. If conditions (17) *and* (18) *are unfulfilled, then the coefficient is zero. If* (17) *and* (18) *are fulfilled, but not* (19), *then* (16) *is still valid provided that it is calculated on the basis of that restricted set of indices r corresponding to non-zero $n_{r\cdot}$.*

Proof. The result is a trivial generalization of one proved earlier in (3), so that I shall merely sketch the proof here.

The coefficient $C_{jk}(n_{rs})$ equals the coefficient of $\prod_{rs} (\theta b_{rs})^{n_{rs}}$ in the jkth element of the matrix

$$\sum_0^\infty (\theta B)^\nu = (I-\theta B)^{-1} = \frac{\text{adj}(I-\theta B)}{|I-\theta B|}. \tag{20}$$

Now, we know by Theorem 1 that, provided that θ is small enough, the coefficient of $\prod_{rs} (\theta b_{rs})^{m_{rs}}$ in $|I-\theta B|^{-1}$ is equal to the same coefficient in the absolute term of

$$\prod_j \left(1-\theta \sum_k b_{jk} z_k/z_j\right)^{-1},$$

and this is

$$\frac{\prod_r m_{r\cdot}!}{\prod_{rs} m_{rs}!}, \tag{21}$$

if $m_{r\cdot} = m_{\cdot r}$ $(r = 1, 2, \ldots, p)$, and is zero otherwise. Knowing this, it is a direct matter to obtain the result of the theorem by piecing together the corresponding coefficient in the expansion of $\Delta_{jk}|I-\theta B|^{-1}$, where

$$(\Delta_{jk}) = \text{adj}(I-\theta B). \tag{22}$$

4. The coefficient of $\theta^N \prod_r z_r^0$ in

$$\Delta_{jk} \prod_r \left(1-\theta \sum_s b_{rs} z_s/z_r\right)^{-1}$$

is the jkth term of B^N, and so for large N is asymptotic to $\lambda^N \alpha_j \beta_k$, where λ is the eigenvalue of B with greatest modulus (if such a single value exists) and (α_j), (β_j) are the corresponding right-hand and left-hand eigenvectors. I shall quote a somewhat more general result, which is an immediate generalization of a result previously proved in (3).

THEOREM 3. *Assume that the $p \times p$ matrix B has an eigenvalue λ whose modulus is greater than those of the remaining eigenvalues or of any of the eigenvalues of the principal minors of B. Then the coefficient of $\theta^N \prod_r z_r^{\nu_r}$ in the expansion of*

$$\Delta_{jk} \prod_r \left(1-\theta \sum_s b_{rs} z_s/z_r\right)^{-1} \tag{23}$$

on $|z_1| = |z_2| = \ldots = |z_p|$ when θ is small, is

$$\lambda^N \alpha_j \beta_k \prod_r \alpha_r^{\nu_r} + O(N^{pk+p-k-2}\mu^N) \tag{24}$$

if $\sum \nu_r = 0$, and is otherwise zero, where μ is a number whose modulus is smaller than that of λ and

$$k = \tfrac{1}{2} \sum_r |\nu_r|. \tag{25}$$

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